Exercises for the PhD course Graph Theory

Lecture 6

1. (a) A k-chromatic graph is called *critical* if $\chi(G - v) < \chi(G)$ for every $v \in V(G)$. Show that every k-chromatic graph has a critical k-chromatic induced subgraph, and that any such subgraph has minimum degree at least $k - 1$.

   (b) Find a graph G for which Brooks' theorem yields a significantly weaker bound than Proposition 5.2.2.

2. Let G be a k-colorable graph, and let P be a set of vertices in G such that $d(x, y) \geq 4$ whenever $x, y \in P$. Prove that every $(k + 1)$-coloring of P extends to a proper $(k + 1)$-coloring of G.

3. Show that the following statements are equivalent for a graph G.

   (i) $\chi(G) \leq k$;

   (ii) G has an orientation without directed paths of length k;

   (iii) G has an acyclic such orientation (i.e. such an orientation with no directed cycles).