MAI0199 Computational Linear Algebra

# Computer Exercise 3. Sylvester's Law of Inertia

## 1 General information

The assignment consists of a mixture of theoretical exercises and practical programming. At the end of the exercise a written report with well structured solutions to the theoretical questions and also Matlab programs, and graphs or plots that summarize the computational results should be sent by email to `Fredrik.Berntsson@liu.se`. In order to reduce the number of Matlab programs keep old code as comments when you modify a program in an exercise.

## 2 Background and Theory

A symmetric matrix $A \in \mathbb{R}^{n\times n}$ can be reduced to tridiagonal form using an orthogonal matrix,

$$T = QAQ^T,$$

where $Q$ can be written as a product of Householder reflections. Thus instead of studying the eigenvalues of a general symmetric matrix it is sufficient to consider only tridiagonal symmetric matrices.

In this execise we will formulate a method for finding bounds for the eigenvalues of a symmetric matrix, or even computing a few of the eigenvalues, by taking advantage of the fact that it is inexpensive to compute the $LU$, or $LDL^T$, decomposition of tridiagonal matrices.

The method is based on *Sylvesters Law of Inertia* that allows us to compute eigenvalues by a bisection procedure called *spectrum slicing*. The algorithm is based on the following:

**Definition** The *Inertia* of a symmetric matrix $A$ is a triplet $(m, z, p)$ where $m$, $z$, and $p$, are the number of positive, zero, and negative eigenvalues respectively. □

**Theorem** If $A \in \mathbb{R}^{n\times n}$ is symmetric and $X \in \mathbb{R}^{n\times n}$ is non-singular then $A$ and $X^TAX$ has the same inertia. □

## 3 Spectrum Slicing

Let,

$$A = \begin{pmatrix} 2 & -1 & & & & \\ -1 & 3 & -1 & & & \\ & -1 & 4 & -1 & & \\ & & -1 & 5 & -1 & \\ & & & -1 & 6 & -1 \\ & & & & -1 & 7 \end{pmatrix}.$$

**Exercise 3.1** Consider the matrix $A$ above. Compute the decomposition $A = LDL^T$ by using the Matlab routine `ldl`. Does the result imply that $A$ only has positive eigenvalues? □

**Remark** In Matlab `[L,D]=ldl(A)` calculates an $LDL^T$ decomposition but $D$ may be block diagonal with a few $2 \times 2$ blocks on the diagonal. This is not a problem since the eigenvalues of $2 \times 2$ matrices are known explicitly.

By using *Sylvesters Law of Inertia* together with a shift we can count the number of eigenvalues of $A$ that are bigger than a number $\sigma$. We do this by computing the decomposition

$$A - \sigma I = LDL^T,$$

where pivoting is avoided completely. Since pivoting is not used the decomposition can fail to exist. This is not a problem however since we can simply shift $\sigma$ a little and try again. It can be shown that for this particular case large elements in the $L$ matrix does not cause any inaccuracy in the results.

**Exercise 3.2** If pivoting is used then the routine `ldl` computes a decomposition $P^T AP = LDL^T$. Can you still conclude that the matrices $A$ and $D$ has the same inertia? □

**Exercise 3.3** Use a shifts $\sigma = 1$ and $\sigma = 8$ to prove that all the eigenvalues of $A$ are located in the interval $1 < \lambda < 8$. Additionally use the shift $\sigma = 2$ to prove that exactly one eigenvalue belongs to the interval $1 < \lambda < 2$. □

**Remark** It is possible to implement this type of procedure very efficiently to compute all eigenvalues of a tridiagonal matrix. For matrices with special structure you can sometimes use congruence transformations $B = XAX^T$ to obtain simpler matrices for the purpose of proving positive or negative definite, or even to prove that no eigenvalue exists within certain intervals.