MAI0199 Computational Linear Algebra

Computer Exercise 4.5. The Conjugate Gradient Method

1 General information

The assignment consists of a mixture of theoretical exercises and practical programming. At the end of the exercise a written report with well structured solutions to the theoretical questions and also Matlab programs, and graphs or plots that summarize the computational results should be sent by email to `Fredrik.Berntsson@liu.se`. In order to reduce the number of Matlab programs keep old code as comments when you modify a program in an exercise.

2 The Conjugate Gradient Method

The *Conjugate gradient method* is as follows:

$r^{(0)} = b - Ax^{(0)}$, $p_0 := r^{(0)}$.

`for` $j = 1, 2, \ldots$ `do`

$\alpha_j = (r^{(j)}, r^{(j)})/(Ap_j, p_j)$.
$x^{(j+1)} := x^{(j)} + \alpha_j p_j$.
$r^{(j+1)} := r^{(j)} - \alpha_j A p_j$.
$\beta_j := (r^{(j+1)}, r^{(j+1)})/(r^{(j)}, r^{(j)})$.
$p_{j+1} := r_{j+1} + \beta_j p_j$.

`end`

Provided that the matrix A is symmetric and positive definite then the above algorithm converges to the solution of the linear system $Ax = b$.

Exercise 2.1 Write a Matlab function that implements the above formulas. The function should be used as follows:

```
>>[ x , Residuals ]=ConjugateGradient( A , b , x0 , MaxIter , tol );
```

where `Residuals` is a vector of the residuals during the iterations. The stopping criteria should be based on the relative residual $\|r\|_2 \leq \texttt{tol}\|b\|_2$. $\square$

Exercise 2.2 Load the `bundle1` test problem in Matlab and calculate the dimension and number of non-zeros for the matrix A.

Calculate the LU decomposition of A. How many non-zeros are there in the L and U matrices? □

The above shows that an iterative solver can potentially be alot more efficient than a direct one.

Exercise 2.3 Use your function to solve the linear system $Ax = b$; using $x_0 = 0$. Terminate the iterations when the relative residual is smaller than tol $= 10^{-10}$. How many iterations are needed? □

Hint The starting residual is quite large hence the relatively large number of iterations. Plot the residuals using `semilogy` to clearly see the convergence speed.