MAI0119 Computational Linear Algebra

Fredrik Berntsson, Email: `fredrik.berntsson@liu.se`

Problems: Basic Theory, Linear Systems and Least Squaers. Complete exercises for a minimum of 10p to pass.

(1p) **1.** An $n \times n$ matrix is said to be *elementary* if it can be written as $A = I - uv^T$, where u and v are vectors. What condition on u and v ensures that A is non-singular? Also prove that A^{-1} is also elementary (by showing $A^{-1} = I - \alpha uv^T$).

(1p) **2.** Let $v \in \mathbb{R}^n$. Prove or disprove

$$\|v\|_1 \|v\|_\infty \leq \frac{1+\sqrt{n}}{2} \|v\|_2.$$

(1p) **3.** Let B be any submatrix of A. Show that $\|B\|_p \leq \|A\|_p$.

(1p) **4.** Suppose $u \in \mathbb{R}^m$ and $v \in \mathbb{R}^n$. Show that if $E = uv^T$ then $\|E\|_F = \|E\|_2 = \|u\|_2 \|v\|_2$ and that $\|E\|_\infty \leq \|u\|_\infty \|v\|_1$.

(2p) **5.** Let $A \in \mathbb{R}^{m \times n}$. Prove the relation $\frac{1}{\sqrt{m}} \|A\|_1 \leq \|A\|_2 \leq \sqrt{n} \|A\|_1$.

(1p) **6.** Suppose $A \in \mathbb{R}^{m \times m}$ and $B \in \mathbb{R}^{m \times n}$, $m > n$. How many operations are required to evaluate the formula $z = (I + A + A^2)Bx + y$, where x and y are vectors.

(1p) **7.** Prove the *Sherman-Morrison* formula

$$(A - uv^T)^{-1} = A^{-1} + A^{-1}u(1 - v^T A^{-1} u)^{-1} v^T A^{-1}.$$

(1p) **8.** Suppose we have two vectors $a_1, a_2 \in \mathbb{R}^m$ and we want to compute a projection of a vector x onto $\text{span}(a_1, a_2)^\perp$. The QR decomposition of the matrix (a_1, a_2) is available. Give a formula for the desired projection $y = Px$.

(1p) **9.** Suppose we have a linear system $Ax = b$ where

$$A = \begin{pmatrix} 2 & 1 & -2 \\ -1 & 0 & 3 \\ 1 & 2 & -1 \end{pmatrix} \text{ and } b = \begin{pmatrix} 6 \\ 1 \\ -3 \end{pmatrix}.$$

During the first step of Gaussian elimination we multiply the system with a matrix L_1 such that the new system $L_1Ax = L_1b$ is

$$\begin{pmatrix} 2 & 1 & -2 \\ 0 & 0.5 & 2 \\ 0 & 1.5 & 0 \end{pmatrix} x = \begin{pmatrix} 6 \\ 4 \\ -6 \end{pmatrix}.$$

Give the Gausstransformation L_1.

(1p) **10.** Let $r = b - A\widehat{x}$ be the residual for an approximate solution to the linear system $Ax = b$. Prove the formula:

$$\|x - \widehat{x}\| \leq \|A^{-1}\|\|r\|.$$

(2p) **11.** Suppose we divide a matrix A into 4 blocks

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

Let $A_{11} = L_{11}U_{11}$ be the LU decomposition of the first block. Show how the steps of block Gaussian elimination are carried out so that the LU decomposition of the whole matrix A is obtained. For simplicity you may assume that pivoting is not needed.

(1p) **12.** How would you solve a partitioned linear system

$$\begin{pmatrix} L_1 & 0 \\ B & L_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

where L_1 and L_2 are lower triangular. Show the steps in terms of the given submatrices and vectors.

(1p) **13.** If A is both an ortgohonal matrix and an orthogonal projection. What can you conclude about A?

(1p) **14.** Suppose $A \in \mathbb{R}^{m \times n}$, $m > n$, and that we have the QR decomposition

$$A = Q \begin{pmatrix} R \\ 0 \end{pmatrix} = Q_1R$$

Show that $P = I - Q_1Q_1^T$ is an orthogonal projection onto Range$(A)^{\perp}$. Also show that $Pb = r = b - Ax$ where x is the least squares solution.

(1p) **15.** Prove that the product of two lower triangular matrices is also lower triangular and that the inverse of a lower triangular matrix is also lower triangular.

(2p) **16.** Suppose B is an $n \times n$ matrix and assume B is both orthogonal and triangular. Prove that B is a diagonal matrix and that the diagonal entries are ± 1. Use the result to prove that the decomposition $A = QR$ is "essentially unique", i.e. only the sign of the diagonal entries in R may differ.

(2p) **17.** Let

$$A = \begin{pmatrix} R & w \\ 0 & v \end{pmatrix} \text{ and } b = \begin{pmatrix} c \\ d \end{pmatrix},$$

where R is $k \times k$, w, c is $k \times 1$, and v, d is $(m-k) \times 1$. Show that if A has full column rank then $\min \|Ax - b\|_2^2 = \|d\|_2^2 - (v^T d / \|v\|_2)^2$.