MAI0119 Computational Linear Algebra

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Problems: Error analysis and Eigenvalues. Complete exercises for a minimum of 15p to pass.

(1p) **1.** Assume the existance of a square root function satisfying $\mathrm{fl}(\sqrt{x}) = \sqrt{x}(1+\epsilon)$, with $|\epsilon| \leq u$. Give en algorithm for computing $\|x\|_2$ and bound the round–off effors.

Can cancellation occur here?

(2p) **2.** Show that if $E \in \mathbb{R}^{m\times n}$, with $m \geq n$, then $\||E|\|_2 \leq \sqrt{n}\|E\|_2$. This is useful for deriving norm bounds from absolute value bounds.

(1p) **3.** Suppose A is symmetric and positive definite. Let M_1 be the Guass transformation such that

$$M_1 A = \begin{pmatrix} a_{11} & w^T \\ 0 & \tilde{A}_{22} \end{pmatrix}$$

Show that

$$M_1 A M_1^T = \begin{pmatrix} a_{11} & 0^T \\ 0 & B \end{pmatrix}$$

where B is symmetric and positive definite.

(1p) **4.** A Householder reflection is a matrix of the form $H = I - 2vv^T$, where $\|v\|_2 = 1$. Prove that

$$\mathrm{fl}(HA) = HA + F$$

and give a bound for F. Also show that if H is orthogonal this is equivalent to $\mathrm{fl}(HA) = H(A+E)$, $\|F\|_2 = \|E\|_2$ so applying a Reflection is backwards stable.

(1p) **5.** Show that if

$$R = \begin{pmatrix} R_{11} & R_{12} \\ 0 & R_{22} \end{pmatrix}$$

is normal and $\lambda(R_{11}) \cap \lambda(R_{22}) = \emptyset$ then $R_{12} = 0$.

(1p) **6.** Show that if S is skew-symmetric (i.e. $S^T = -S$) then $Q = (I+S)(I-S)^{-1}$ is orthogonal. The matrix Q is called the *Caley transform* of S.

(1p) **7.** Suppose (λ, x) is a known eigenpair for the matrix A. Give an algorithm for computing an orthogonal matrix P such that

$$P^T A P = \begin{pmatrix} \lambda & w^T \\ 0 & \widetilde{A} \end{pmatrix}.$$

Hint: Compute P as a product of Givens rotations.

(1p) **8.** Prove that all the eigenvalues of a Hermitean matrix A are real. Use the result to prove that the eigenvalues of a real symmetric matrix are real.

(1p) **9.** Prove that for any matrix norm induced by a vector norm the spectral radius satisfies $\rho(A) \leq \|A\|$.

(1p) **10.** A matrix A is said to be *nilpotent* if $A^k = 0$ for some positive integer k. Show that all eigenvalues of a nilpotent matrix are zero. Can you conclude that A is the zero matrix?

(1p) **11.** Suppose $A, B \in \mathbb{R}^{n \times n}$ and in addition A is non-singular. Show that AB and BA have the same eigenvalues.

(1p) **12.** Show that for any two real vectors u and v the formula

$$\det(I + uv^T) = 1 + u^T v$$

holds.

(1p) **13.** Let $A \in \mathbb{R}^{n \times n}$ and $\rho(A) < 1$. Show that $I - A$ is non-singular and

$$(I - A)^{-1} = \sum_{k=0}^{\infty} A^k.$$

(1p) **14.** Let A be real and symmetric with eigenvalues $\lambda_{min} \leq \lambda(A) \leq \lambda_{max}$. Show that for $x \neq 0$,

$$\lambda_{min} \leq \frac{x^T A x}{x^T x} \leq \lambda_{max}.$$

(1p) **15.** Suppose $A \in \mathbb{R}^{n \times n}$, $\operatorname{rank}(A) = n$, and $A = QR$, where Q is orthogonal and R is upper triangular.

a) Show that RQ is a Hessenberg matrix if A is a Hessenberg matrix.

b) Show that RQ is tridiagonal if A is symmetric and tridiagonal.

Hint: Write Q as a product of Givens rotations. It is enough to treat the 4×4 case to clearly see the pattern.

(1p) **16.** Given $A \in \mathcal{C}^{n\times n}$ show that for every $\varepsilon > 0$ there exists a diagonalizable matrix B such that $\|A - B\|_2 \leq \epsilon$. This shows that the set of diagonalizable matrices is dense in $\mathcal{C}^{n\times n}$.

Hint Use the Schur decomposition.

(1p) **17.** Suppose (λ, x) is a known eigenpair for the matrix A. Give an algorithm for computing an orthogonal matrix P such that

$$P^T A P = \begin{pmatrix} \lambda & w^T \\ 0 & \widetilde{A} \end{pmatrix}$$

Write P as a product of Givens rotations.

(1p) **18.** Show that if $A \in \mathbb{R}^{m\times n}$ then $\|A\|_F \leq \sqrt{\text{rank}(A)}\|A\|_2$.

(2p) **19.** Suppose,

$$A = \begin{pmatrix} D & v \\ v^T & d_n \end{pmatrix}$$

where $D = \text{diag}(d_1, d_2, \ldots, d_{n-1})$ has distinct diagonal entries and $v \in \mathbb{R}^{n-1}$ has no non-zero elements. Show that if $\lambda \in \lambda(A)$ then $D - \lambda I$ is non-singular, and that λ is a root of

$$f(\lambda) = \lambda + \sum_{k=1}^{n-1} \frac{v_k^2}{d_k - \lambda} - d_n.$$

(2p) **20.** With the same assumptions as in the Bauer-Fike theorem show that

$$\min_{\lambda\in\lambda(A)} |\lambda - \mu| \leq \| |X^{-1}||E||X| \|_p.$$