MAI0102 ENUMERATIVE COMBINATORICS, 2018
PROBLEM SET 1

(Before beginning, please read the homework policy on the course webpage.)

Each problem is worth ten points. You may get partial credit for non-useless, non-perfect solutions. The problems are not ordered by difficulty. Hand in your solutions no later than October 11.

1. We have claimed that differentiation in the ring $\mathbb{C}[[x]]$ of formal power series satisfies (among other things) the product rule $(FG)' = F'G + FG'$. Prove it.

2. How many k-tuples $(S_1, \dots, S_k)$ satisfy $S_1 \subseteq S_2 \subseteq \cdots \subseteq S_k \subseteq [n]$?

3. Recall that a polynomial is called *homogeneous* if all its monomials are of the same degree. For example, $2xyz + 3x^2z - z^3$ is homogeneous of degree 3 whereas $2xyz + 3x^2z - z$ is inhomogeneous. Together with 0, the homogeneous polynomials of degree k in n variables (with coefficients in $\mathbb{R}$, say) form a vector space (over $\mathbb{R}$) which we call V_k^n. Find $\dim(V_k^n)$ and a simple expression for the *Hilbert series* $\sum_{k \geq 0} \dim(V_k^n) x^k$.

4. (This is Exercise 1.7 in Stanley.) Let the numbers $f(i)$ be defined by

$$e^{x+\frac{x^2}{2}} = \sum_{n \geq 0} f(n) \frac{x^n}{n!}.$$

Find a simple expression for the sum

$$\sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} f(i).$$

Hint. Consider $e^{\frac{x^2}{2}} = e^{x+\frac{x^2}{2}} \cdot e^{-x}$.

5. Let $f(n)$ be the number of complete matchings on $[2n]$, i.e. the number of graphs on vertex set $[2n]$ where every vertex is incident to exactly one edge. Observe (or, define) $f(0) = 1$. Find simple formulae for $f(n)$ and for the exponential generating function $\sum_{n \geq 0} f(n) \frac{x^n}{n!}$.