MAI0102 ENUMERATIVE COMBINATORICS, 2018
PROBLEM SET 2

Each problem is worth ten points. You may get partial credit for non-useless, non-perfect solutions. The problems are not ordered by difficulty. Hand in your solutions no later than October 25.

1. Let $a(n)$ be the number of compositions[1] of n into parts of size 2 or 3. For example, $9 = 3+3+3 = 3+2+2+2 = 2+3+2+2 = 2+2+3+2 = 2+2+2+3$, so that $a(9) = 5$. Express the generating function $F(x) = \sum_{n\geq 0} a(n)x^n$ as a rational function.
Hint. Find a recurrence formula first.

2. (This is Exercise 1.44a in Stanley.) Show that the total number of cycles of all even[2] permutations in $\mathfrak{S}_n$ and the total number of cycles of all odd permutations in $\mathfrak{S}_n$ differ by $(-1)^n(n-2)!$.
Hint. Generating functions for Stirling numbers of the 1st kind.

3. Let k and n be positive integers. Prove that the following are all equal:

- The number of partitions of n where each part occurs at most $2k-1$ times.
- The number of partitions of n with no part divisible by $2k$.
- The number of partitions of n where parts that are divisible by k occur at most once each.

4. Give a combinatorial proof of Stanley's equation (2.14), the recurrence formula $D(n) = (n-1)(D(n-1) + D(n-2))$ for the derangement numbers.
Hint. Either n is in a 2-cycle or n is not in a 2-cycle.

5. Let $M(m,n)$ be the number of $m \times n$-matrices with entries in $\{0,1\}$ such that every row and every column contains at least one 1. Prove that

$$M(m,n) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} (2^{m-k}-1)^n.$$

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[1] Recall that a *composition* of n is a way to express n as a sum of positive integers, where the order of the terms (or, parts) matters.

[2] Recall that a permutation $\pi \in \mathfrak{S}_n$ with k cycles is *even* if $(-1)^{n-k} = 1$. Otherwise, π is *odd*.