MAI0102 ENUMERATIVE COMBINATORICS, 2018
PROBLEM SET 3

Each problem is worth ten points. You may get partial credit for non-useless, non-perfect solutions. The problems are not ordered by difficulty. Hand in your solutions no later than November 8.

1. Give a direct proof of the identity $N_0 = \sum_k (-1)^k r_k (n-k)!$ (formula (2.23) in Stanley) using the principle of inclusion-exclusion.

2. Let $f_d(m,k)$ be the number of ways to choose k points from a collection of m distinguishable points arranged in a circle if each pair of chosen points must be separated by at least d points. Show that

$$f_d(m,k) = \frac{m}{m-kd}\binom{m-kd}{k}.$$

3. Let $f(n)$ be the number of permutations $\pi \in \mathfrak{S}_{2n}$ such that $i - n \leq \pi(i) \leq n + i$ for all $i \in [2n]$. Show that

$$f(n) = \sum_{k=0}^{2n-2} (-1)^k (2n-k)! \sum_{i=0}^{k} S(n, n-i) S(n, n-k+i).$$

Comment. It is possible to simplify the above formula considerably. Thus, if you happen to arrive at a simpler formula for $f(n)$, I will accept that too. But don't cook up even more complicated formulae, please.

4. Let k and n be positive integers. Define a matrix M of size $n \times n$ with $\binom{k+i}{k+j-i}$ being the entry on row i, column j. Prove that all minors of M are nonnegative.
Comment. Recall that $\binom{a}{b} = 0$ if $b < 0$.
Hint. Consider lattice paths with starting points and end points on two appropriately chosen straight lines.

5. A *plane partition which fits in an $n \times n \times n$ cube* is an $n \times n$ array of integers between 0 and n such that all rows and columns are weakly decreasing. Show that the number of such plane partitions is $\det(M_n)$, where M_n is the $n \times n$ matrix whose entry in position (i,j) is $\binom{2n}{n+i-j}$.
Hint. Stare at the following picture:

$\begin{matrix} 2 & 1 \\ 2 & 0 \end{matrix}$ $\leftrightarrow$