MAI0102 ENUMERATIVE COMBINATORICS, 2018
PROBLEM SET 4

Each problem is worth ten points. You may get partial credit for non-useless, non-perfect solutions. The problems are not ordered by difficulty. Hand in your solutions no later than November 22.

1. Let L be a (not necessarily finite) lattice. Recall that L is called *complete* if every subset $S \subseteq L$ has a join and a meet.[1] Suppose there exists an integer $\alpha \in \mathbb{N}$ such that $|C| < \alpha$ whenever $C \subseteq L$ is a chain. Prove that L is complete.

2. Let L be a finite geometric lattice. A set $A \subset L$ of atoms is called *independent* if $|A|$ is the rank of $\vee A$. Now suppose $A, B \subset L$ are two independent atom sets such that $|A| > |B|$. Show that there exists some $a \in A \setminus B$ such that $B \cup \{a\}$ is independent.

3. (This is essentially Stanley's Exercise 3.127a.) Prove that the number of maximal chains in the partition lattice Π_n is $\frac{n!(n-1)!}{2^{n-1}}$.

4. We are given a finite poset P and want to partition it into antichains. Show that the minimum number of antichains required is the same as the maximum cardinality of a chain in P.

5. (This is related to Stanley's Exercise 3.34.) Find two nonisomorphic posets P such that $J(P)$ has rank-generating function $F(J(P), x) = (1+x)(1+x^2)(1+x+x^2)$.

[1] Clearly, finite lattices are always complete.