# MAI0102 ENUMERATIVE COMBINATORICS, 2018
# PROBLEM SET 6
# *—CORRECTED VERSION—*

Each problem is worth ten points. You may get partial credit for non-useless, non-perfect solutions. The problems are not ordered by difficulty. Hand in your solutions no later than December 20.

**1.** Given positive integers $m$ and $k$, define a poset $P$ on the set $([m] \times [k]) \cup \{\hat{0}, \hat{1}\}$ by declaring that $\hat{0}$ is the minimum, $\hat{1}$ is the maximum, and $(x_1, y_1) < (x_2, y_2)$ in $P$ if and only if $x_1 < x_2$ in the usual total order on integers. (In other words, $P \setminus \{\hat{0}, \hat{1}\}$ is the $m$-fold ordinal sum of $k$-element antichains.) Prove that the Möbius function of $P$ satisfies $\mu(\hat{0}, \hat{1}) = -(1-k)^m$.

**2.** (This and the next problem are essentially Stanley's Exercise 3.129.) Let $P$ be a finite poset equipped with a minimum $\hat{0}$ and a maximum $\hat{1}$. Suppose $P$ has an automorphism $f$ of order $p$, where $p$ is a prime, such that the only fixed points of $f$ are $\hat{0}$ and $\hat{1}$. Show that $\mu(\hat{0}, \hat{1}) \equiv -1 \pmod{p}$, where $\mu$ is the Möbius function of $P$.

**3.** Recall that Wilson's theorem states that $(p-1)! \equiv -1 \pmod{p}$ for any prime $p$. Prove Wilson's theorem by applying the result of the previous problem to the partition lattice $\Pi_p$.
*Hint. Permutations of* $[n]$ *provide automorphisms of* $\Pi_n$.

**4.** Suppose $P$ is a finite poset whose Möbius function satisfies that $\mu(x, y)$ is even for every $x < y$ such that $y$ does not cover $x$. Show that every closed interval $[a, b]$, $a < b$, of $P$ has an odd number of atoms.[1]

**5.** For an integer $n \geq 2$, consider the poset of all simple[2] disconnected graphs on vertex set $[n]$ ordered by inclusion of edge sets. Notice that the edgeless graph is the minimum $\hat{0}$. Add an artificial maximum element $\hat{1}$ and call the resulting poset $G_n$. Prove that $\mu(\hat{0}, \hat{1}) = (-1)^{n-1}(n-1)!$, where $\mu$ is the Möbius function of $G_n$.
*Hint. There is a natural closure operator on* $G_n$ *whose image is isomorphic to the partition lattice.*

[1] Recall that an *atom* of a poset with a minimum element $a$ is an element covering $a$.
[2] That is, no loops or multiple edges are allowed.