MAI0106 Nov-Dec 2011

Assignment 1

Consider the hyperbolic equation

$$u_t = Pu, \quad P = \frac{\partial}{\partial x}a + a\frac{\partial}{\partial x}, \; , \quad 0 \le x \le 1\,, \quad 0 \le t \le 0.5\,, \quad a(x) > 0\,, \tag{1}$$

Let Q be a semibounded difference operator satisfying

$$(v, Qv)_h \le 0 \tag{2}$$

for all v satisfying certain boundary conditions. The so called θ-scheme is defined by

$$(I - \theta k Q)v^{n+1} = (I + (1-\theta)kQ)v^n\,. \tag{3}$$

1. Construct homogeneous (zero data) boundary conditions and a scalar product such that P is semibounded ($(u, Pu) \le 0$). Specify where the boundary condition is imposed.

2. Consider equation (1), construct discrete boundary conditions and a scalar product such that $Qv = D_0(av) + aD_0v$ is semibounded and satisfy (2).

3. Compute the eigenvalues of Q and plot them for N = 20, 40, 80. What is the minimal real part of all the eigenvalues for each N ?

4. Run the θ-scheme for θ = 1/2, 3/4, 1, for N = 20, 40, 80, and demonstrate the stronger damping with increasing θ. Use

$$a(x) = 0.5 - 0.1x(1-x)\,,$$
$$u(x,0) = 1 + \sin(10\pi(x - 0.25))\,,$$

   and $k = 2h$.

5. **Challenge:** Prove that (3) is unconditionally stable for $\frac{1}{2} \le \theta \le 1$, and that the norm has a stronger decrease with time for larger θ in this interval (which you should have seen in task 4. above).

Motivate your answers clearly !