MAI0106 Nov-Dec 2011

Assignment 3

As **the first task**, consider the IBVP problem

$$\begin{aligned} u_t + Au_x + Bu_y &= \epsilon(Cu_{xx} + Du_{yy}) + F(x,y,t) \\ Lu(0,y,t) &= g(y,t) \\ u(x,y,0) &= f(x,y) \end{aligned} \tag{1}$$

where $x \geq 0,\ 0 \leq y \leq 2\pi,\ \epsilon > 0$. The solution is periodic in y, $u(x > \bar{x}, y, t) = 0$ and

$$A = B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

As **the second task**, consider the scalar advection equation

$$\begin{aligned} u_t &= -u_x + \theta u, \quad x \geq 0, \quad t \geq 0 \\ u(0,t) &= g_1(t) \quad (a) \\ u_x(0,t) &= g_2(t) \quad (b) \\ u(x,0) &= f(x), \quad x \geq 0. \end{aligned} \tag{2}$$

1. Use the Laplace transform technique *and* the energy-method in (1) to determine how many boundary conditions should be imposed at $x = 0$.
2. Determine the boundary operator L in (1) such that the problem is well posed. Give as many examples of L that you can find.
3. Derive an energy estimate for (2) using boundary condition (a). Can a positive θ lead to a well-posed problem?
4. Use the Laplace-transform technique and compute the continuous spectrum for boundary condition (b) and $\theta = -1$.
5. Discretise (2) on the domain $0 \leq x \leq 1$ using central differences (D_0) in the interiour and the approximations below at the boundaries. Eq. (3) is the numerical outflow approximation. Eq. (4) and (5) both approximate the PDE at the inflow boundary using (2)(b).

   Compute the spectrum for the two approximations using the Matrix technique as discussed in the lecture notes and compare with the result in 4. Discuss the result.

$$\begin{aligned} (u_N)_t &= -(u_N - u_{N-1})/h + \theta u_N, & (3) \\ (u_0)_t &= -(u_1 - u_0)/h - u_0/h + (u_0)_t/(\theta h) + \theta u_0, & (4) \\ (u_0)_t &= \theta u_0. & (5) \end{aligned}$$

Motivate your answers clearly !