Assignment 5

As **the first task**, consider the coupling of the two scalar advection equations

$$
\begin{aligned}
u_t + au_x &= 0, \ -1 \le x \le 0 \\
v_t + av_x &= 0, \ 0 \le x \le 1 \\
u(0,t) &= v(0,t).
\end{aligned} \tag{1}
$$

Let $u_a = \sin(2\pi(x-t))$ be the exact periodic solution.

As **the second task**, consider the scalar wave propagation problem,

$$
\begin{aligned}
u_t + au_x + bu_y &= 0, \quad (x,y) \in \Omega, \\
Lu &= g(x,y,t), \quad (x,y) \in \delta\Omega \\
u(x,y,0) &= f(x,y), \quad (x,y) \in \Omega.
\end{aligned} \tag{2}
$$

The wave propagation direction $\bar{a} = (a,b)$ is constant and both a and b are postive.

1. Introduce a mesh and write up the semi-discrete formulation of problem (1) using summation-by-parts operators and the SAT-penalty formulation for the boundary and interface conditions. Do it for different operators and meshes on the domains.

2. Prove stability of the semi-discrete formulation for (1) using the energy-method (determine the penalty parameters). This means that both the left boundary treatment and the interface must be stable.

3. In the 1st calculation use your scheme above and $u_a(-1,t)$ as boundary data at the left boundary. In the 2nd calculation, write a scheme using periodic boundary conditions and no interface. Use $u_a(x,0)$ as the initial condition. Use the 4th order operators on page 312 in C-R. For the periodic case modify accordingly. Integrate with classical explicit R-K in time. Show by calculations what accuracy you have in space for both schemes. Next, run both schemes to $t = 100$, do mesh refinement, plot the L_2 error as a function of time and discuss the result.

4. Let $\Omega = [0,1] \times [0,1]$ be the unit square. Use the energy-method on (2) to determine the boundary operator L and where to impose boundary conditions.

5. Discretize (2) using high order finite difference methods (FDM) on SBP form and use penalty terms for the boundary condition. The approximation will look like,

   $$U_t + a(P_x^{-1}Q_x \otimes I_y)U + b(I_x \otimes P_y^{-1}Q_y)U = (P_x^{-1} \otimes P_y^{-1})((E_0 \otimes \Sigma_x) + (\Sigma_y \otimes E_0))(U - G).$$

   $(E_0)_{11}$ is one, the rest of $(E_0)_{ij}$ is zero. Use the energy method and determine Σ_x and Σ_y so that the approximation is stable. Assume that P_x and P_y are diagonal.

Motivate your answers clearly !