Exercises II

Consider the scalar initial value problem

$$u_t = au_x\,, \quad t \geq 0\,,$$
$$u(x,0) = f(x)\,,$$

where a is a real constant and $f(x)$ is 2π-periodic.

1. Investigate the stability for the leap-frog scheme

$$u_j^{n+1} = u_j^{n-1} + 2kaD_0u_j^n\,.$$

   What restriction on the time step is imposed by the von Neumann condition? What is the stability condition?

2. The leap-frog scheme requires data at two time levels t_0 and t_1. Suggest a method for obtaining u_j^1.

3. Prove that the norm of the solutions to $u_t = au_x - u$ is non-increasing with time. Find out if this is true also for the solutions u_j^n to the leap-frog scheme with aD_0 substituted by $aD_0 - I$.

4. Derive the stability condition for the schemes

$$u_j^{n+1} = u_j^n + kaD_+u_j^n\,.$$

   and

$$u_j^{n+1} = u_j^n + kaD_-u_j^n\,.$$

5. Prove that the Crank-Nicholson scheme

$$u_j^{n+1} = u_j^n + \frac{ka}{2}D_0(u_j^{n+1} + u_j^n)$$

   is unconditionally stable.

6. Prove that the Crank-Nicholson scheme for $u_t = au_{xx}$ (D_0 substituted by D_+D_-) is unconditionally stable if $a > 0$.