Lecture 4

Initial Boundary Value Problems

Boundary Conditions, where?
how many? what form?
data? Energy + Laplace method.

1)

Well-posedness requires:

1. The right number of b.c.
2. The right place for the b.c.
3. The right form of – " –

Accuracy require:

4. Correct data for the form in 3.
5. Non-refl. (Gunilla)

Steady-state calculations require

6. $u_t = Au$, $Re(\lambda_A) < -\delta$, $\delta > 0$

Error-bounded scheme require

7. b.c. giving dissipation.

2)

Simple examples where and how many?

$u_t + a u_x = 0 \; ; \; 0 \leq x \leq 1 \; ; \; t \geq 0$

1. Physical method (not to rec.)

$u = u^0(x - at)$

b.c. $x = 0$ if $a > 0$

b.c $x = 1$ if $a < 0$

2. Energy-method (simplest one)

$\|u\|_t^2 = a u_{x=0}^2 - a u_{x=1}^2$

b.c $x = 0$ if $a > 0$

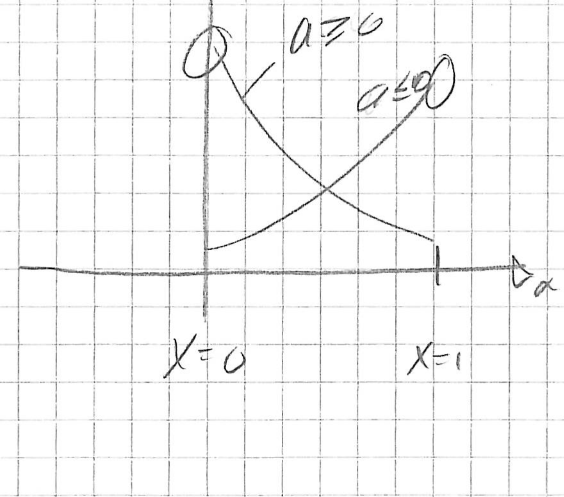
b.c $x = 1$ if $a < 0$ (Maximally Semibounded)

3. Laplace transform (most general)

$s\hat{u} + a\hat{u}_x = 0 \Rightarrow \hat{u} = \alpha e^{-\frac{s}{a}x}, \; Re(s) > 0$

b.c $x = 0$ if $a < 0$

b.c $x = 1$ if $a < 0$



3) Simple examples (where and how many?)

$u_t = \varepsilon u_{xx}$; $0 \le x \le 1$; $t \ge 0$

1. Physical method ? (heat energy above ?)

2. Energy-method

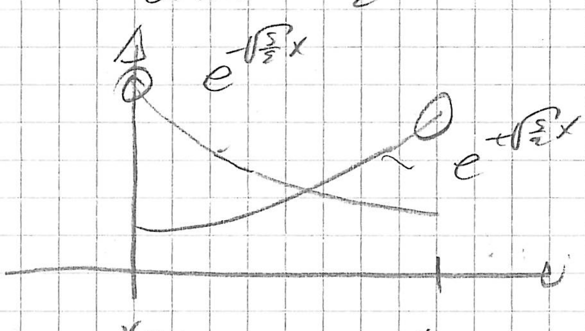
$\|u\|_t^2 + 2\varepsilon\|u_x\|^2 = 2\varepsilon u u_x\Big|_0^1 = \varepsilon \begin{pmatrix} u \\ u_x \end{pmatrix}^T \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{A} \begin{pmatrix} u \\ u_x \end{pmatrix}\Bigg|_0^1$

$\lambda_{1,2} = \pm 1$

Always one positive eigenvalue $\Rightarrow$ 1 b.c at $x=0,1$.

3. Laplace-transform

$\hat{u}_{xx} = \frac{s}{\varepsilon}\hat{u} \Rightarrow \hat{u} = \alpha_1 e^{-\sqrt{\frac{s}{\varepsilon}}x} + \alpha_2 e^{+\sqrt{\frac{s}{\varepsilon}}x}$; $Re(s) > 0$



One b.c. at each boundary, one decaying solution from each boundary.

4)

Simple examples (where and how many?)

$u_t + a u_x = \varepsilon u_{xx}$, $0 \le x \le 1$, $t \ge 0$

1. Physical method ?

2. Energy-method

$\|u\|_t^2 + 2\varepsilon \|u_x\|^2 = (-a u^2 + 2\varepsilon u u_x)\Big|_0^1 = \begin{pmatrix} u \\ u_x \end{pmatrix}^T \begin{pmatrix} -a & +\varepsilon \\ +\varepsilon & 0 \end{pmatrix} \begin{pmatrix} u \\ u_x \end{pmatrix}\Bigg|_0^1$

$\lambda_{1,2} = -\frac{a}{2} \pm \sqrt{\frac{a}{2} + \varepsilon^2}$ $\lambda_1 > 0, \lambda_2 < 0$ A

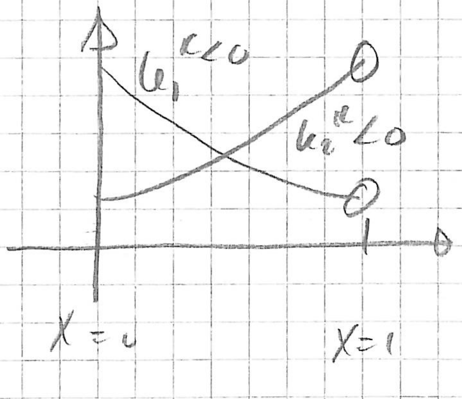
Alway one positive eigenvalue => 1 b.c. at $x=0,1$

3 Laplace-transform

$u_{xx} - \frac{a}{\varepsilon} u_x - \frac{s}{\varepsilon} u = 0$ $u = e^{kx}$ =>

$k^2 - \frac{a}{\varepsilon} k - \frac{s}{\varepsilon} = 0$ $Re(s) > 0$

$k_{1,2} = \frac{a}{\varepsilon} \pm \sqrt{\left(\frac{a}{2\varepsilon}\right)^2 + \frac{s}{\varepsilon}}$ $k_1^R > 0$ $k_2^R < 0$ =>



$\hat{u} = \alpha_1 e^{k_1 x} + \alpha_2 e^{k_2 x}$

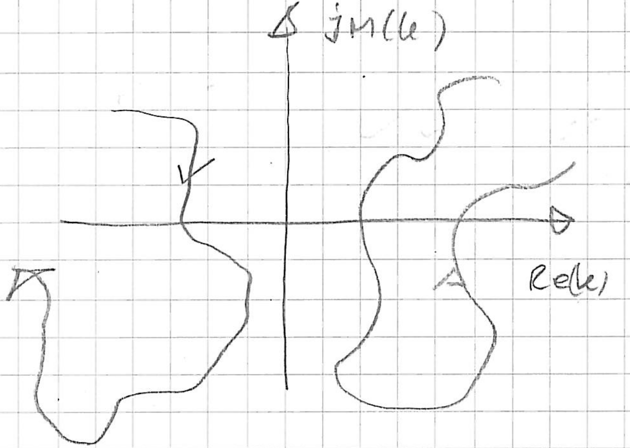
5)

$$k^2 - \frac{a}{\varepsilon}k - \frac{s}{\varepsilon} = f(k,s) = 0 \quad (*)$$

$$f(ip,s) = -p^2 - \frac{a}{\varepsilon}ip - \frac{s}{\varepsilon} = 0 \Rightarrow$$

$$s = -\varepsilon p^2 + iap \quad Re(s) < 0$$

$\therefore$ $Re(s) > 0 \Rightarrow$ no $k^2 = 0$



eigenvalues not on imaginary axis

$\therefore$ Same nr k, with $Re(k) < 0$

if $Re(s) \to \infty$.

Ansats $k = C_0 s^{\alpha} + C_1 s^{\alpha-\delta}$, $\alpha, \delta > 0$

into $(*)$ $\Rightarrow$

$$\left(C_0^2 s^{2\alpha} + C_0 C_1 s^{2\alpha-\delta} + \ldots\right) - \frac{a}{\varepsilon}\left(C_0 s^{\alpha} + C_1 s^{\alpha-\delta} + \ldots\right) - \frac{s}{\varepsilon} = 0$$

$\Rightarrow$

6) Balance $\Rightarrow$

$$2\alpha = 1 \quad \therefore \quad \alpha = 1/2$$
$$2\alpha = 2\alpha - \delta \qquad C_0^2 = \frac{1}{\varepsilon}$$
$$2\alpha = \alpha$$
$$2\alpha = \alpha - \delta$$

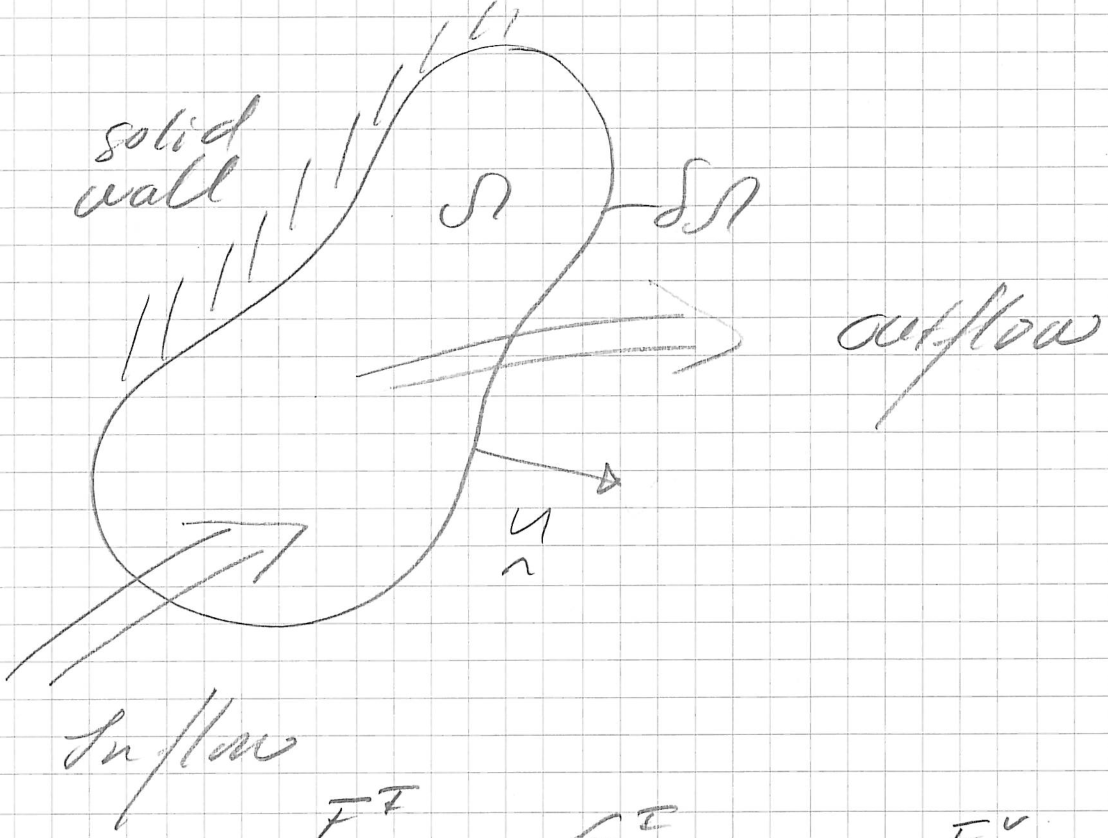
$$\therefore \quad u = \pm\sqrt{\frac{\delta}{\varepsilon}} + O(\delta^{1/2-\delta})$$

Always one positive an one negative

$\Rightarrow$ 1 bc at $x = 0, 1$.

<u>General procedure for all systems</u>

7) <u>The energy-method for pr, position and form of boundary conditions</u>



$$U_t + (\overset{F^I}{AU})_x + (\overset{G^I}{BU})_y = \varepsilon[(\overset{F^V}{C_{11}U_x + C_{12}U_y})_x + (\overset{G^V}{C_{21}U_x + C_{22}U_y})_y$$

$$U_t + (F^I - \varepsilon F^V)_x + (G^I - \varepsilon G^V)_y = R$$

<u>Conservation law</u> (R = source term)

Conservs: mass, momentum, energy Euler - N-S

– " – Electric and Magnetic fields Maxwell.

– " – ......

A, B, C_{ij} = Constant Symmetric Matrices

8) Energy :

$$\iint_\Omega u^T u_t \, d\Omega + \iint_\Omega u^T F^I_x + u^T L^I_x \, dx\,dy = \varepsilon \iint_\Omega u^T F^u_x + u^T L^u_y \, dx\,dy$$

$$\Rightarrow$$

$$\frac{1}{2}\|u\|^2_t + \iint_\Omega \left(\frac{u^T A u}{2}\right)_x + \left(\frac{u^T B u}{2}\right)_y d\Omega =$$

$$\varepsilon \iint_\Omega (u^T F^u)_x + (u^T L^u)_x - (u_x^T F^u + u_y^T L^u) \, d\Omega$$

$$\Rightarrow$$

$$\|u\|^2_t + \oint_{\partial\Omega} u^T A u \, dy - u^T B u \, dx = \oint_{\partial\Omega} (u^T F^u) dy - (u^T L^u) dx$$

$$-2\varepsilon \iint_\Omega u_x^T F^u + u_y^T L^u \, d\Omega .$$

$$\|u\|^2_t = \underbrace{-\oint_{\partial\Omega} (u^T A u - 2\varepsilon u^T F^u) dy - (u^T B u - 2\varepsilon u^T G^u) dx}_{BT}$$

$$\underbrace{-2\varepsilon \oint_\Omega (u_x^T F^u + u_x^T L^u) \, d\Omega}_{DI}$$

9)

$$BT = -\oint_{\partial\Omega} u^T(A\,dy - B\,dx)u - 2\varepsilon u^T(F^u dy - G^u dx)$$

$$= -\oint_{\partial\Omega} u^T\underbrace{(Au, Bu)}_{\underset{\sim}{F}^I}\cdot \underset{\sim}{n} - 2\varepsilon u^T\underbrace{(F^u, G^u)}_{\underset{\sim}{F}^u}\cdot \underset{\sim}{n}\, ds$$

$$= \oint_{\partial\Omega} u^T \underset{\sim}{F}^u \cdot \underset{\sim}{n} - 2\varepsilon u^T \underset{\sim}{F}^u \cdot \underset{\sim}{n}\, ds =$$

$$= -\oint_{\Omega} u^T(\underset{\sim}{F}^I - 2\varepsilon \underset{\sim}{F}^u)\cdot \underset{\sim}{n}\, ds$$

$$DI = -2\varepsilon \iint_{\Omega} \begin{pmatrix} u_x \\ u_y \end{pmatrix}^T \underbrace{\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}} \begin{pmatrix} u_x \\ u_y \end{pmatrix} d\Omega$$

Must be neg. semi-def.

Boundary Conditions ?

where, how many, what form?

10)

$$BT = -\oint_{\Omega} u^T \underbrace{A\,dy - B\,dx}_{\tilde{A}\,ds} u - 2\varepsilon u^T (F^u dy - G^u dx) =$$

$$-\oint_{\Omega} u^T \hat{A} u - 2\varepsilon u^T (\hat{C}_x u_x + \hat{C}_y u_y)\, ds =$$

$$\left(\hat{A} = (A, B) \cdot \underset{\sim}{n}, \quad \hat{C}_x = (C_{11}, C_{21}) \cdot \underset{\sim}{n} \right.$$

$$\left. \hat{C}_y = (C_{12}, C_{22}) \cdot \underset{\sim}{n} \right)$$

$$-\oint_{\Omega} \begin{pmatrix} u \\ \varepsilon u_x \\ \varepsilon u_y \end{pmatrix}^T \underbrace{\begin{pmatrix} \hat{A} & \hat{C}_x & \hat{C}_y \\ \hat{C}_x & 0 & 0 \\ \hat{C}_y & 0 & 0 \end{pmatrix}}_{\hat{\hat{A}}} \begin{pmatrix} u \\ \varepsilon u_x \\ \varepsilon u_y \end{pmatrix} ds$$

Can rotate $\hat{\hat{A}}$

Nr of b.c's = nr of neg eigenvalues of $\hat{\hat{A}}$

Where? on all parts of boundary, check all gridpoints

form of b.c's = characteristic variables corresponding to neg. eigenvalues.

See Nordström & Svärd 2005 SINUM.

11)

Some times Characteristic B.C.
very complicated form and
no data exist.

- find nr and guess form

Ex: $u_t + (au - \varepsilon u_x)_x = 0$ { We know 1 b.c. at each boundary

$$BT = au^2 - 2\varepsilon u u_x = a(u^2 - 2\tfrac{\varepsilon}{a} u u_x) =$$

$$a\left(\left(u - \tfrac{\varepsilon}{a}u_x\right)^2 - \left(\tfrac{\varepsilon}{a}u_x\right)^2\right) =$$

$$a^{-1}\left(\underbrace{(au - \varepsilon u_x)^2}_{\text{Total flux}} - \underbrace{(\varepsilon u_x)^2}_{\text{Viscous flux}}\right)$$

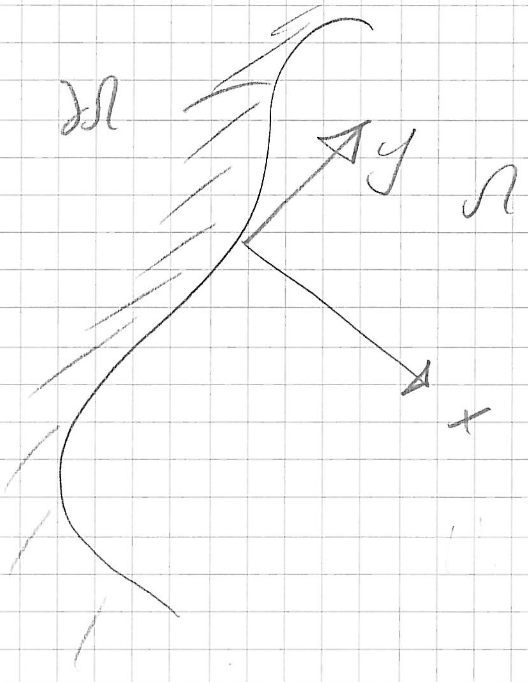
$$\|u\|^2 = a^{-1}\left((au-\varepsilon u_x)^2 - (\varepsilon u_x)^2\right) - 2\varepsilon \|u_x\|^2$$

$a > 0 \qquad au - \varepsilon u_x = g$

$a < 0 \qquad -\varepsilon u_x = g$

See Nordstrom, Comp & fluid 95

12) <u>The Laplace-transform technique for nr and position of boundary conditions.</u>



$$U_t + (F^I - \varepsilon F^v)_x + (G^I - \varepsilon G^v)_y = 0$$

$$F^I = AU \quad G^I = BU$$

$$F^v = C_{11}U_x + C_{12}U_y$$

$$G^v = C_{21}U_x + C_{22}U_y$$

All matrices symmetric and constant.

Laplace in t and Fourier in y $\Rightarrow$

$$s\hat{U} + A\hat{U}_x + i\omega B\hat{U} = \varepsilon\left[C_{11}U_{xx} + C_{12}i\omega\hat{U}_x + C_{21}i\omega U_x - \omega^2 C_{22}U\right]$$

$$(s + i\omega B + \varepsilon\omega^2 C_{22})\hat{U} + [A - i\varepsilon\omega(C_{12}+C_{21})]\hat{U}_x - \varepsilon C_{11}\hat{U}_{xx} = 0$$

Ansatz $\hat{U} = \psi e^{\frac{k}{s}x} \Rightarrow$

13)

$$\Big[\underbrace{(s\Sigma + i\varepsilon\omega B + (\varepsilon\omega)^2 C_{22}) + [A - i(\varepsilon\omega)(C_{12}+C_{21})]k - C_{11}k^2}_{C(s,\omega)}\Big]\psi = 0$$

$f(k,s,\omega) = \det[C(k,s,\omega)] = 0 \Rightarrow$ eigenvalues k.

<u>Theorem:</u> No k on imaginary axis for $Re(s) > 0$

Show that no eigenvalue $k = i\beta$ for $Re(s) > 0$. I.e. no imaginary spatial eigenvalue on imaginary axis.

<u>Proof:</u> insert $k = i\beta$ multiply from the left with ψ^* $\Rightarrow$

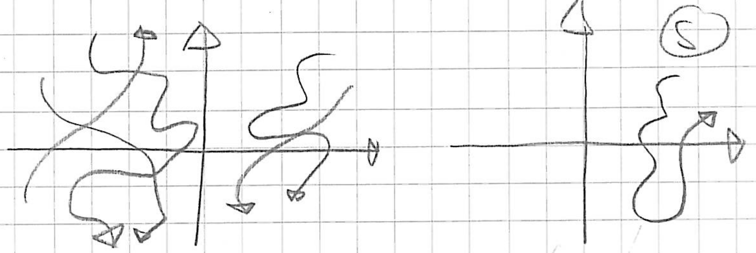
$$Real(\psi^* C(s, i\beta)\psi) = \hat{n}|\psi|^2 + \begin{pmatrix}\beta\psi \\ \varepsilon\omega\psi\end{pmatrix}^* \underbrace{\begin{pmatrix} C_{11} & \frac{C_{12}+C_{21}}{2} \\ \frac{C_{12}+C_{21}}{2} & C_{22}\end{pmatrix}}_{\geq 0 \text{ since Diss}} \begin{pmatrix}\beta\psi \\ \varepsilon\omega\psi\end{pmatrix} = 0$$

$\Rightarrow \psi > 0$ not possible for $k = i\beta$

14)

∴ Eigenvalues k split into 2

groups. No crossing "to other side".



No of boundary conditions totally given by the polynomial degree of k in

$$\det(C(k,\omega,s)) = f(k,\omega,s) = 0$$

Sign of real part of k obtained by

Ansatz $k = c_0 s^{\alpha} + c_1 s^{\alpha-\delta} + \ldots \quad \alpha, \delta > 0$

for $\mathrm{Re}(s) \to \infty$.

Nr of boundary conditions at $x=0$

is same as nr of b.c. with

$\mathrm{Re}(k) < 0$.

See Nordström 95 JCP.

15)

Steady-State Calculations, Small reflections and Error Bounded solutions

Ex: $u_t + a u_x = \varepsilon u_{xx} + F$

$$u(x,0) = f$$

$$Lu = 0$$

Energy $\Rightarrow$

$$\|u\|_t^2 + 2\varepsilon\|u_x\|^2 = a u^2 - 2\varepsilon u u_x \Big|_0^1 + 2(u,F)$$

$$\|u\|_t^2 + 2\varepsilon\|u_x\|^2 = a^{-1}\left((au-\varepsilon u_x)^2 - (\varepsilon u_x)^2\right) + 2(u,F)$$

$$Lu_{x=0} = au - \varepsilon u_x = 0$$

$$\Rightarrow$$

$$Lu_{x=1} = -\varepsilon u_x = 0$$

$$2\|u\|\|u\|_t + 2\varepsilon\|u_x\|^2 \leq -(\varepsilon u_x)^2/a\big|_{x=0} - a u^2_{x=1} + 2\|u\|\|F\|$$

$$\|u\|_t \leq -\frac{\varepsilon\|u_x\|^2}{\|u\|} - \frac{\left((\varepsilon u_x)^2/a\right)_{x=0} + a u^2_{x=1}}{\|u\|} + \|F\|$$

$$\|u\|_t \leq -(\alpha+\beta)\|u\|^2 + \|F\|^2$$

16)

$$\alpha = -\varepsilon \frac{\|u_x\|^2}{\|u\|}, \quad \beta = -\frac{(\varepsilon u_x)(a+au)'}{\|u\|^2}$$

Note that $\alpha = 0$ but $\beta \neq 0$ for $\varepsilon \to 0$!

Time integration assuming that $(\alpha + \beta) = \eta = \text{const}$.

$$\left(\|u\| e^{+\eta t}\right)' \leq e^{\eta t} \|F\| \Rightarrow$$

$$\|u\| e^{+\eta t} \leq \|f\| + \int_0^t e^{+\eta \xi} \|F\| d\xi$$

$$\|u\| \leq e^{-\eta t} \|f\| + e^{-\eta t} \int_0^t e^{\eta \xi} \|F\| d\xi \leq$$

$$\leq e^{-\eta t} \|f\| + \|F\|_{max} e^{-\eta t} \int_0^t e^{\eta \xi} d\xi$$

$$\leq e^{-\eta t} \|f\| + \|F\|_{max} e^{-\eta t} \left(\frac{e^{\eta t} - 1}{\eta}\right)$$

$$\leq e^{-\eta t} \|f\| + \|F\|_{max} \left(\frac{1 - e^{-\eta t}}{\eta}\right)$$

17) <u>Steady-state</u> (see Nordström 1989 JCP)

- Influence of initial data disappear

<u>Error Bound</u> (see Nordström 2007 SISC)

$$\text{Error max} = \frac{\|F\|}{\eta} < \infty$$

<u>Small-reflection</u> (see Nordström 1995) 1995

BT maximally dissipative.