CME 326 SPRING 2011

The Continuous and Discrete spectrum

References

Scalar: "Spectral analysis ...."
Attached tech. report.

Systems: "The influence of"
Attached article.

1)

The Continuous and Discrete Spectrum

1. The Notation in previous notes are used without a complete explanation.

2. We will compute the spectral content of the continuous and discrete problem. They should be similar.

3. This is hard to find in any book but the best way to find out what your discretisation does to your problem.

2)

<u>Continuous problem.</u>

$$u_t = Pu + F; \quad 0 \le x \le 1, \; t > 0$$

$$Lu = g; \quad x = 0, 1, \; t > 0 \quad (1)$$

$$u = f; \quad 0 \le x \le 1, \; t = 0$$

$P = \sum A_\nu \partial^\nu / \partial x^\nu$, L = boundary op.

Laplace transform for $F = g = f = 0$.

$$(sI - P)\hat{u} = 0; \quad 0 \le x \le 1$$
$$L\hat{u} = 0 \quad x = 0, 1 \quad (2)$$

Ansatz $\varphi = e^{kx} \Rightarrow$

$$(sI - \sum A_\nu k^\nu)\varphi = 0 \quad (3)$$

Non-trivial solution if

$$|sI - \sum A_\nu k^\nu| = 0 \quad (4)$$

3)

(4) $\Rightarrow$ k_i , k_i into (3) $\Rightarrow$

ψ_i and finally

$$\hat{u}(x) = \sum_i \alpha_i \psi_i e^{(k_i x)} \qquad (5)$$

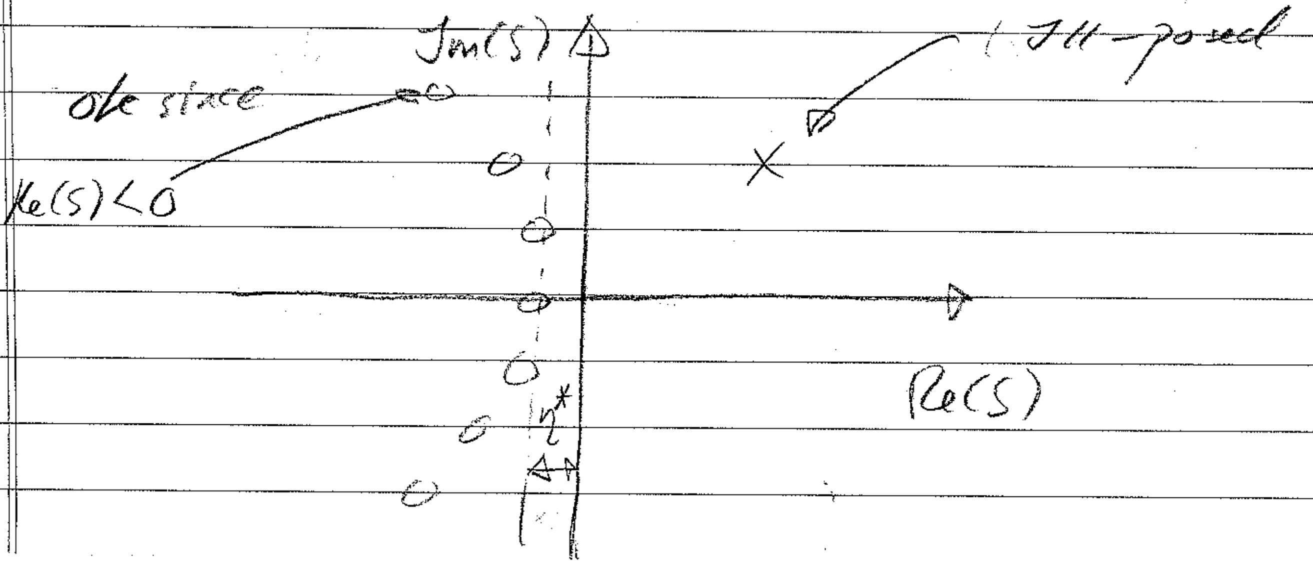
Boundary Conditions $L\hat{u} = 0 \Rightarrow$

$$C(s)\, \underline{\alpha} = 0 \qquad (6)$$

Spectrum = Set of eigenvalues

s that makes

$$\det(C(s)) = 0 \qquad (7)$$

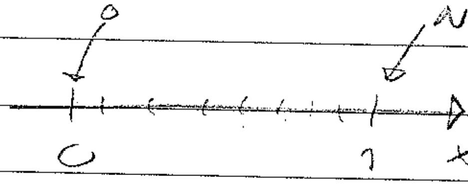


4) <u>Note</u> The solution decays / converges to steady-state if not forced by data as

$$u \sim e^{\varphi^* t} \qquad (8)$$

<u>Note</u> eigenvalues, and φ^* moves depending on how you chose your <u>boundary operator L</u> since L constructs C'.

Discrete problem.



$$U_{jt} = QU_j + F_j \qquad j=1..N-1$$

$$L_h U = g \qquad j=0,N \qquad (1)$$

$$U_j = f_j \qquad j=0 \ldots N$$

Laplace transform for $F=g=f$.

$$(\hat{s}I - kQ)\hat{U}_j = 0 \qquad j=1..N-1 \qquad (2)$$

$$L_h \hat{U} = 0 \qquad j=0,N$$

Ansatz $\hat{U}_j = \psi \kappa^j \Rightarrow$

$$\left(\hat{s}I - \sum A_\nu \kappa^\nu\right)\psi = 0 \qquad (3)$$

Non-trivial solution if

$$\left|\hat{s}I - \sum A_\nu \kappa^\nu\right| = 0 \qquad (4)$$

6)

(4) $\Rightarrow k_i$, k_i into (3) $\Rightarrow$

φ_i and finally

$$\hat{u}_j = \sum_i \alpha_i \varphi_i(k_i)^j \qquad (5)$$

Boundary Conditions $L_u \hat{u} = 0$

$\Rightarrow$

$$C(\tilde{s})\, \underline{\alpha} = 0 \qquad (6)$$

Spectrum = set of eigen values

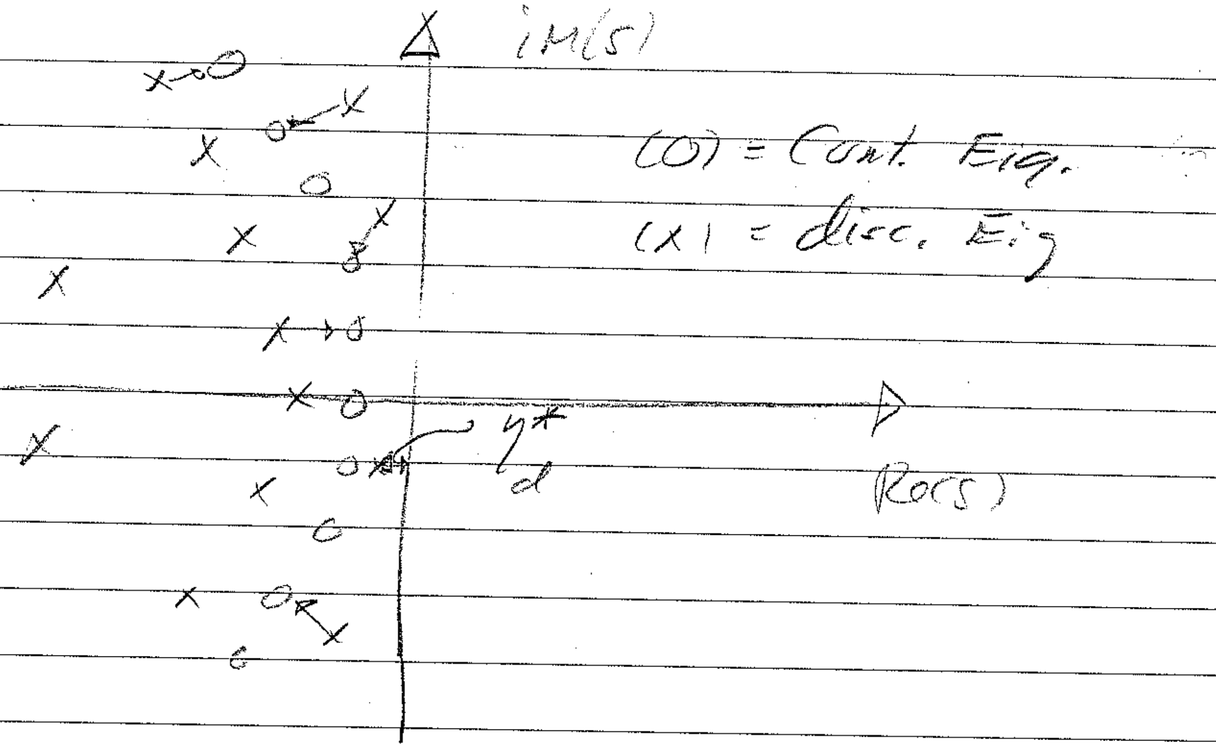
s that makes

$$\det(C(\tilde{s})) = 0 \qquad (7)$$

Note! $s = \tilde{s}/h$

post processing.

7)



<u>Note</u> Some (x) converge to (o)'s, some not. As $\Delta x \to 0$ more and more (x) → (o) but even more are created.

<u>Note</u> The solution decays / converges to steady-state if not forced as

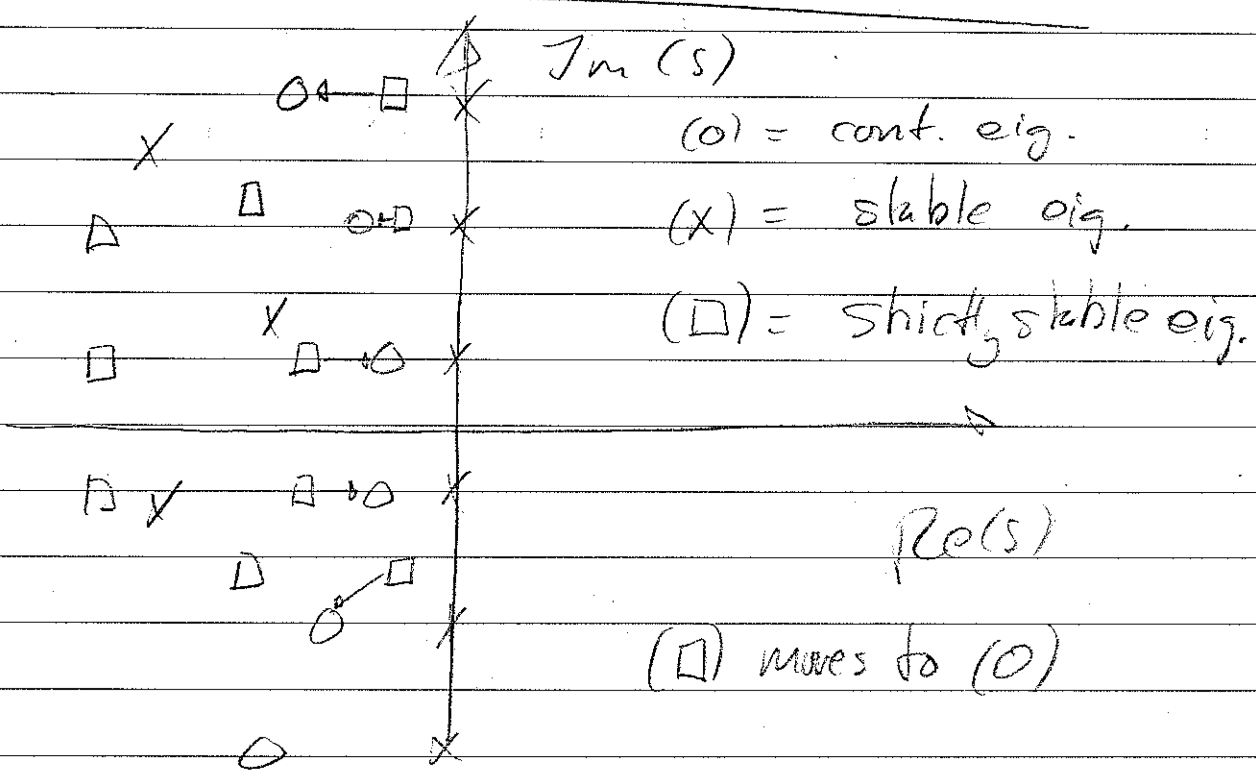
$$u_j \sim e^{\eta_d^* t}$$

8)

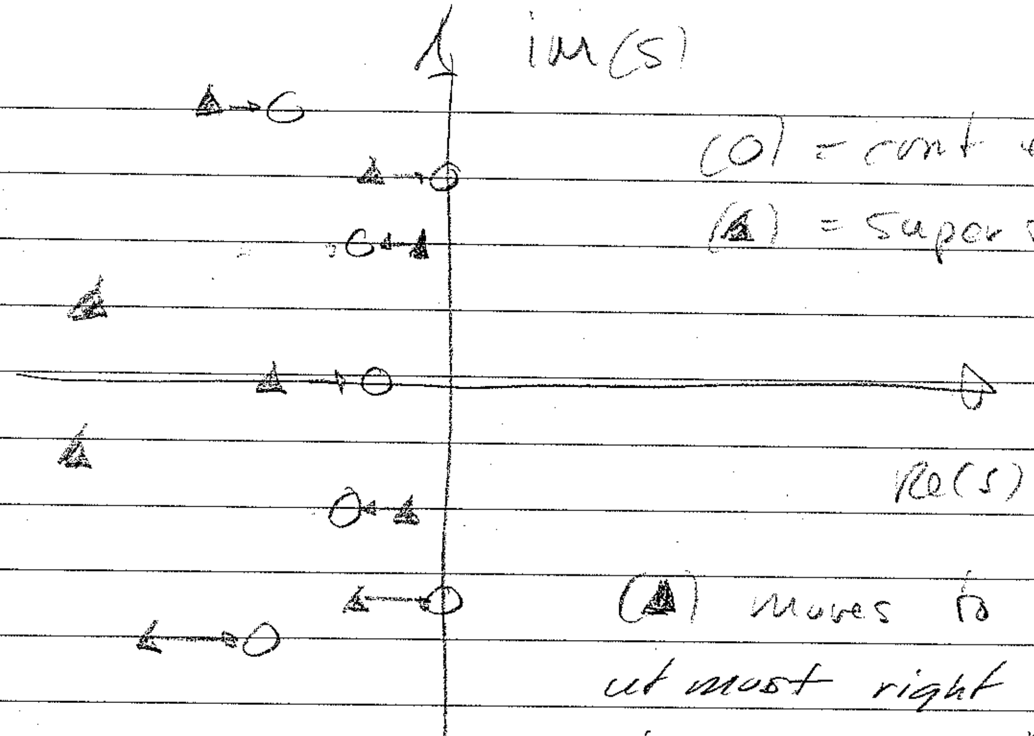
Note eigen values, and y_d^* moves depending on how you choose your $\underline{L_h}$ which now consist of $\underline{L + \text{Numerical B.C.}}$.

and constructs $\underline{C}$

Stability and Strict Stability



9)



(O) = cont eig.
(▲) = super strict stb.

(▲) moves to the utmost right lying eigenvalue (γ^*) from the right side.
"Correct landing approach"

In practise it is easier and leads to the same result for the discrete eigenvalues if you compute your eigenvalues directly as

$$\left| \underbrace{(uQ + L_u)}_{\circ} - \hat{s} I \right| = 0$$

Combined internal opp and B.C

Use a standard eigen solver.
Must treat both boundaries at same time

9.5 Matrix form

$\underset{\sim}{u}_t = h\hat{Q}\underset{\sim}{u}$ where

$\hat{Q} = hQ + h_{?} = Q + B.C.$

Laplace-transform ⇒

$(\hat{Q} - \tilde{s}I)\hat{u} = 0$

∴ $\tilde{s}$ = eigenvalues to $\hat{Q}$

+ Easy to use

− Must treat both boundaries together.

To compare the continuous and discrete spectrum is:

- Best method for finding if your scheme is correct.

- Best method to study convergence to steady-state

- Use B.C. to push the spectrum where you want to have it.

  - steady-state $\Rightarrow$ far left
  - wave-propagation $\Rightarrow$ on imag axis
  - . . .