10

THE LAPLACE TRANSFORM METHOD FOR INITIAL-BOUNDARY-VALUE PROBLEMS

10.1 SOLUTION OF HYPERBOLIC SYSTEMS

Initial-boundary-value problems for systems with constant coefficients can be solved using the Laplace transform. As we will see, the Laplace transform method is often the only way we can decide on the stability of a given finite difference scheme. We only use elementary properties of the Laplace transform (see Appendix A.2).

We consider the quarter-space problem for the system

$$\frac{\partial}{\partial t}\begin{bmatrix} u^I \\ u^{II} \end{bmatrix} = A\frac{\partial}{\partial x}\begin{bmatrix} u^I \\ u^{II} \end{bmatrix} + F, \qquad 0 \le x < \infty, \quad t \ge 0, \tag{10.1.1a}$$

where A is diagonal and

$$A = \begin{bmatrix} \Lambda^I & \\ & \Lambda^{II} \end{bmatrix}, \qquad \Lambda^I > 0, \quad \Lambda^{II} < 0.$$

We prescribe initial data

$$u(x,0) = f(x), \tag{10.1.1b}$$

and boundary conditions

$$\|u(\cdot,t)\| < \infty \text{ for every fixed } t. \tag{10.1.1c}$$

where L^I and L^{II} are constant matrices. We assume that F, f, and g are smooth functions with compact support.

We know already that the above problem is well posed if, and only if, L^{II} is nonsingular. However, we arrive at the same conclusion using the Laplace transform. We start with the following lemma.

Lemma 10.1.1. *Consider the initial-boundary-value problem (10.1.1) with* $F \equiv g \equiv 0$. *It is not well posed if we can find a complex number* s *with* $\text{Re}\, s > 0$ *and initial values* $f(x)$ *with* $0 < \|f\| < \infty$ *such that*

$$w(x,t) = e^{st}f(x), \qquad \text{Re}\, s > 0 \tag{10.1.2}$$

is a solution.

Proof. Assume that there is a solution of the above type. Define the sequence $\{f_j(x)\}_{j=1}^{\infty}$ by

$$f_j(x) = \frac{f(jx)}{\|f(jx)\|}, \quad \text{i.e.,} \quad \|f_j\| = 1.$$

Then

$$w_j(x,t) = e^{jst}f_j(x),$$

are also solutions satisfying

$$\|w_j(\cdot,t)\| = e^{j(\text{Re}\, s)t}, \qquad j = 1,2,\ldots.$$

Therefore, the problem cannot be well-posed because we can construct solutions that grow arbitrarily fast. This proves the lemma.

REMARK. If Eq. (10.1.1) had contained lower order terms, then solutions w with $\text{Re}\, s \le \eta_0$ are permissible.

We will now give conditions guaranteeing that solutions of the type of Eq. (10.1.2) exist.

Theorem 10.1.1. *A solution of the type of Eq. (10.1.2) exists; that is, the initial-boundary-value problem is not well posed, if the eigenvalue problem*

$$s\varphi = A\frac{d\varphi}{dx}, \qquad 0 \le x < \infty, \qquad (10.1.3a)$$

with boundary conditions

$$L^{II}\varphi^{II}(0) + L^{I}\varphi^{I}(0) = 0, \qquad \|\varphi\|^2 < \infty, \qquad (10.1.3b)$$

has an eigenvalue s with $\text{Re}\, s > 0$.

The eigenvalue problem has an eigenvalue s with $\text{Re}\, s > 0$ *if, and only if, L^{II} is singular. Therefore, by our previous result, the initial-boundary-value problem is well posed if, and only if, the eigenvalue problem (10.1.3) has no eigenvalue s with* $\text{Re}\, s > 0$.

Proof. If there is an eigenvalue s with $\text{Re}\, s > 0$, then

$$w(x,t) = e^{st}\varphi(x)$$

is a solution of the type of Eq. (10.1.2). Therefore, the problem is not well posed.

Let us now derive algebraic conditions such that there are no eigenvalues with $\text{Re}\, s > 0$. These conditions are necessary conditions for the problem to be well posed. The general solution of Eq. (10.1.3a) can be written in the form

$$\varphi^{I}(x) = e^{s(\Lambda^{I})^{-1}x}\varphi^{I}(0), \qquad \varphi^{II}(x) = e^{s(\Lambda^{II})^{-1}x}\varphi^{II}(0).$$

For $\|\varphi\| < \infty$, a necessary and sufficient condition is that

$$\varphi^{I}(0) = 0. \qquad (10.1.4)$$

Then the relations (10.1.3b) are satisfied if, and only if,

$$L^{II}\varphi^{II}(0) = 0. \qquad (10.1.5)$$

There are two possibilities.

1. L^{II} *is Nonsingular.* Then the only solution of Eq. (10.1.5) is $\varphi^{II}(0) = 0$, and there is no eigenvalue s with $\text{Re}\, s > 0$. In this case, we know, from our previous results, that the problem is well posed.

fore, there is an eigenvalue s with $\text{Re}\, s > 0$. In fact, all s with $\text{Re}\, s > 0$ are eigenvalues.

This proves the theorem.

Thus, for a well-posed problem, L^{II} must be nonsingular, and, therefore, we can write the boundary conditions in the form

$$u^{II}(0,t) = R^{I}u^{I}(0,t) + g^{II}(t); \qquad (10.1.6)$$

that is, they are necessarily of the form we have discussed earlier.

We can now solve the problem using the Laplace transform. Without restriction, we can assume that $f(x) \equiv 0$. Otherwise, we introduce a new variable $\tilde{u} = u - h(t)f(x)$, $h(0) = 1$, h smooth with compact support. We define the Laplace transform by

$$\hat{u}(x,s) = \int_0^\infty e^{-st}u(x,t)\,dt,$$
$$s = i\xi + \eta, \qquad \xi,\eta \text{ real}, \quad \eta > 0. \qquad (10.1.7)$$

REMARK. From Section 9.2, we know that the solution of Eq. (10.1.1) satisfies an energy estimate. Therefore, the right-hand side of Eq. (10.1.7) is finite for every $\eta > 0$.

By Eq. (10.1.1a)

$$\int_0^\infty e^{-st}u_t\,dt = A\int_0^\infty e^{-st}u_x\,dt + \int_0^\infty e^{-st}F\,dt = A\hat{u}_x + \hat{F}.$$

Therefore,

$$\int_0^\infty e^{-st}u_t\,dt = e^{-st}u\big|_0^\infty + s\int_0^\infty e^{-st}u\,dt$$

implies (observe that $f \equiv 0$)

$$s\hat{u}^{I} = \Lambda^{I}\hat{u}_x^{I} + \hat{F}^{I}, \qquad s\hat{u}^{II} = \Lambda^{II}\hat{u}_x^{II} + \hat{F}^{II}. \qquad (10.1.8a)$$

$$\hat{u}^{II}(0,s) = R^I\hat{u}^I(0,s) + \hat{g}^{II}(s), \qquad \|\hat{u}(\cdot,s)\| < \infty. \tag{10.1.8b}$$

Because the eigenvalue problem (10.1.3) only has a trivial solution, the solution $\hat{u}$ of the homogeneous problem (10.1.8) is identically zero. Thus, the inhomogeneous problem (10.1.8) has a unique solution. We can write down the solution explicitly:

$$\hat{u}^I(x,s) = -\int_\infty^x e^{s(\Lambda^I)^{-1}(x-y)}(\Lambda^I)^{-1}\hat{F}^I(y,s)\,dy,$$
$$\hat{u}^{II}(x,s) = \int_0^x e^{s(\Lambda^{II})^{-1}(x-y)}(\Lambda^{II})^{-1}\hat{F}^{II}(y,s)\,dy + e^{s(\Lambda^{II})^{-1}x}\hat{u}^{II}(0,s), \tag{10.1.9}$$

where $\hat{u}^{II}(0,s)$ is determined by Eq. (10.1.8b).

Using Eq. (10.1.9), we can estimate $\hat{u}(x,s)$ in terms of $\hat{F}$, $\hat{g}^{II}$. However, it is easy to use energy estimates directly. We take the scalar product of Eq. (10.1.8a) with $\hat{u}^I$ and $\hat{u}^{II}$, respectively, and obtain

$$(\hat{u}^I, s\hat{u}^I) + (s\hat{u}^I,\hat{u}^I) = (\hat{u}^I,\Lambda^I\hat{u}^I_x) + (\Lambda^I\hat{u}^I_x,\hat{u}^I) + (\hat{u}^I,\hat{F}^I) + (\hat{F}^I,\hat{u}^I),$$

or

$$\eta\|\hat{u}^I\|^2 = \text{Re}\,(\hat{u}^I,\Lambda^I\hat{u}^I_x) + \text{Re}\,(\hat{u}^I,\hat{F}^I).$$

Integration by parts gives us

$$\text{Re}\,(\hat{u}^I,\Lambda^I\hat{u}^I_x) = -\tfrac{1}{2}\langle\hat{u}^I(0,s),\Lambda^I\hat{u}^I(0,s)\rangle,$$

and, therefore,

$$\eta\|\hat{u}^I\|^2 + \tfrac{1}{2}\langle\hat{u}^I(0,s),\Lambda^I\hat{u}^I(0,s)\rangle \le \|\hat{u}^I\|\,\|\hat{F}^I\|;$$

that is,

$$\|\hat{u}^I\| \le \frac{1}{\eta}\|\hat{F}^I\|, \qquad |\hat{u}^I(0,s)| \le \frac{C}{\eta^{1/2}}\|\hat{F}^I\|.$$

$$\eta\|\hat{u}^{II}\|^2 \le \|\hat{u}^{II}\|\,\|\hat{F}^{II}\| - \frac{1}{2}\langle\hat{u}^{II}(0,s),\Lambda^{II}\hat{u}^{II}(0,s)\rangle,$$
$$\le \frac{\eta}{2}\|\hat{u}^{II}\|^2 + \frac{1}{2\eta}\|\hat{F}^{II}\|^2 - \frac{1}{2}\langle\hat{u}^{II}(0,s),\Lambda^{II}\hat{u}^{II}(0,s)\rangle,$$

or

$$\eta\|\hat{u}^{II}\|^2 \le \text{constant}\left(\frac{1}{\eta}\|\hat{F}^{II}\|^2 + |\hat{u}^{II}(0,s)|^2\right).$$

Using the boundary condition we obtain

$$|\hat{u}^{II}(0,s)|^2 \le \text{constant}\,(|\hat{g}^{II}|^2 + |\hat{u}^I(0,s)|^2) \le \text{constant}\left(|\hat{g}^{II}|^2 + \frac{1}{\eta}\|\hat{F}^I\|^2\right).$$

Therefore,

$$\eta\|\hat{u}\|^2 \le \text{constant}\left(\frac{1}{\eta}\|\hat{F}\|^2 + |\hat{g}^{II}|^2\right), \tag{10.1.10a}$$
$$|\hat{u}(0,s)|^2 \le \text{constant}\left(|\hat{g}^{II}|^2 + \frac{1}{\eta}\|\hat{F}\|^2\right). \tag{10.1.10b}$$

Inverting the Laplace transform gives us the solution of our problem

$$e^{-\eta t}u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi t}\hat{u}(x,i\xi+\eta)\,d\xi,$$

and, by Parseval's relation, Eq. (A.2.17), we obtain, for any $\eta > 0$ and $s = i\xi+\eta$,

$$\int_0^\infty e^{-2\eta t}\|u(\cdot,t)\|^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}\|\hat{u}(\cdot,s)\|^2\,d\xi,$$
$$\le \text{constant}\int_{-\infty}^{\infty}\left(\frac{1}{\eta^2}\|\hat{F}(\cdot,s)\|^2 + \frac{1}{\eta}|\hat{g}^{II}(s)\|^2\right)d\xi,$$
$$= \text{constant}\int_0^\infty e^{-2\eta t}\left(\frac{1}{\eta^2}\|F(\cdot,t)\|^2 + \frac{1}{\eta}|g(t)|^2\right)dt. \tag{10.1.11a}$$

$$\int_0^\infty e^{-2\eta t}|u(0,t)|^2\,dt \le \text{constant} \int_0^\infty e^{-2\eta t}\left(\frac{1}{\eta}\,\|F(\cdot,t)\|^2 + |g(t)|^2\right) dt. \qquad (10.1.11b)$$

Thus, we can estimate the solution in terms of the data. In Section 10.3, we use this estimate to define another concept of well-posedness.

EXERCISES

10.1.1. Assume that $f(x) \neq 0$ in Eq. (10.1.1b). Derive the estimate (10.1.11b) by applying the Laplace transform technique to $\tilde{u} = u - h(t)f(x)$ as described above.

10.2. SOLUTION OF PARABOLIC PROBLEMS

We start with an example. Consider the quarter-space problem for a parabolic system

$$u_t = \Lambda u_{xx} + F, \qquad 0 \le x < \infty, \quad t \ge 0,$$
$$u(x,0) = f(x), \qquad (10.2.1)$$

where

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \qquad \lambda_j = \text{constant} > 0, \quad j = 1,2, \quad u = \begin{bmatrix} u^{(1)} \\ u^{(2)} \end{bmatrix},$$

with boundary conditions

$$u^{(1)}(0,t) = \alpha u^{(2)}(0,t) + g^{(1)}(t),$$
$$u_x^{(2)}(0,t) = \beta u_x^{(1)}(0,t) + g^{(2)}(t),$$
$$\|u(\cdot,t)\| < \infty. \qquad (10.2.2)$$

where α and β are constants. Let us first investigate under what conditions on α and β we can obtain an energy estimate. Here we assume that $g^{(1)} = g^{(2)} = 0$. Integration by parts gives us

$$\frac{d}{dt}\,\|u\|^2 = -2(u_x, \Lambda u_x) - 2B + (u,F) + (F,u),$$

$$B = \mathrm{Re}\,\langle u(0,t), \Lambda u_x(0,t)\rangle = \mathrm{Re}\,((\lambda_1\alpha + \lambda_2\beta)u^{(2)}(0,t)u_x^{(1)}(0,t)).$$

Therefore, to obtain an energy estimate, we need $\lambda_1\alpha + \lambda_2\beta = 0$.

We now investigate the case $\lambda_1\alpha + \lambda_2\beta \neq 0$. As in the previous section, we can prove that the problem is not well posed if the eigenvalue problem corresponding to Eqs. (10.2.1) and (10.2.2)

$$s\varphi = \Lambda\varphi_{xx}, \qquad 0 \le x < \infty, \qquad (10.2.3)$$
$$\varphi^{(1)}(0) = \alpha\varphi^{(2)}(0), \qquad \varphi_x^{(2)}(0) = \beta\varphi_x^{(1)}(0), \qquad \|\varphi\| < \infty, \qquad (10.2.4)$$

has eigenvalues in the right half of the complex plane. In that case, there are solutions that grow arbitrarily fast.

For our example, we have the following theorem:

Theorem 10.2.1. *The problem [Eqs. (10.2.3) and (10.2.4)] has an eigenvalue s with* $\mathrm{Re}\, s > 0$ *if, and only if,*

$$\lambda_2^{-1/2} - \lambda_1^{-1/2}\alpha\beta = 0, \qquad \lambda_j^{-1/2} > 0 \quad j = 1,2. \qquad (10.2.5)$$

Proof. The general solution of Eq. (10.2.3) with $\|\varphi\| < \infty$ for $\mathrm{Re}\, s > 0$ is given by

$$\varphi^{(j)} = e^{-\lambda_j^{-1/2}s^{1/2}x}y^{(j)}, \qquad \mathrm{Re}\,\lambda_j^{-1/2}s^{1/2} > 0,$$
$$y^{(j)} = \text{constant}, \qquad j = 1,2.$$

The boundary conditions are satisfied if, and only if,

$$\begin{bmatrix} 1 & -\alpha \\ -\beta\lambda_1^{-1/2} & \lambda_2^{-1/2} \end{bmatrix} y = 0, \qquad y = \begin{bmatrix} y^{(1)} \\ y^{(2)} \end{bmatrix}. \qquad (10.2.6)$$

Equation (10.2.6) has a nontrivial solution if, and only if, $\lambda_2^{-1/2} - \lambda_1^{-1/2}\alpha\beta = 0$. This proves the theorem.

We now show that if $\lambda_2^{-1/2} - \lambda_1^{-1/2}\alpha\beta \neq 0$, then the problem has a unique solution that can be estimated in terms of the data. We assume again that $f \equiv 0$. The Laplace transform $\hat{u}$ is the solution of

$$s\hat{u} = \Lambda\hat{u}_{xx} + \hat{F},$$
$$\hat{u}^{(1)}(0,s) = \alpha\hat{u}^{(2)}(0,s) + \hat{g}^{(1)}(s),$$

Theorem 10.2.2. *The problem (10.2.11) has a unique solution satisfying an estimate of type (10.2.10), for $\eta > \eta_0$, if, and only if, Eq. (10.2.12) has no eigenvalue s with* $\operatorname{Re} s > 0$.

This theorem is valid for much more general boundary conditions

$$\sum_{j=0}^{p} B_{1j} \frac{\partial^{j+1} u}{\partial t^j \partial x} + B_{0j} \frac{\partial^j u}{\partial t^j} = 0, \quad x = 0. \tag{10.2.13}$$

EXERCISES

10.2.1. Formulate and prove the analogy to Lemma 10.1.1 for parabolic systems

$$u_t = A u_{xx}.$$

10.2.2. Prove Theorem 10.2.2.

10.3. GENERALIZED WELL-POSEDNESS

We again consider the system of partial differential equations

$$\partial u/\partial t = Pu + F, \qquad 0 \le x \le 1, \quad t \ge 0, \tag{10.3.1a}$$

with initial data

$$u(x,0) = f(x), \tag{10.3.1b}$$

and boundary conditions

$$L_0 u(0,t) = g_0(t), \qquad L_1 u(1,t) = g_1(t). \tag{10.3.1c}$$

We assume that the system of differential equations is either symmetric hyperbolic, that is,

$$P = A\frac{\partial}{\partial x} + B, \qquad A = A^*,$$

$$P = A\frac{\partial^2}{\partial x^2} + B\frac{\partial}{\partial x} + C, \qquad A + A^* \ge 2a_0 I > 0.$$

For simplicity, we assume that A, B, and C and the coefficients of L_0 and L_1 are smooth functions of x and t. We also assume that F, f, g_0, and g_1, are smooth bounded functions. To guarantee that the compatibility conditions are satisfied, we assume that F vanishes near the boundary and the initial line and that $f(x)$, $g_0(t)$, and $g_1(t)$ vanish near $x = 0, 1$ and $t = 0$, respectively. Of course, less stringent conditions need be met if only solutions with a fixed number of derivatives are of interest. As before, one can also extend the solution concept to include generalized solutions.

In Section 9.4, we have discussed two different definitions of well-posedness. They differ with respect to the estimate required. All bounds are natural in the context of energy estimates. Unfortunately, energy estimates are not available in many circumstances, and, therefore, other techniques must be used. A very powerful tool is the Laplace transform, which we used in the last two sections.

When using the Laplace transform, it is convenient to assume that $f(x) \equiv 0$. In Section 10.1, how to transform the problem so that this condition is satisfied was explained.

To begin, we assume that the system (10.3.1) has constant coefficients. As in previous sections, we introduce the Laplace transform

$$\hat{u}(x,s) = \int_0^\infty e^{-st} u(x,t)\,dt, \qquad s = i\xi + \eta, \quad \xi, \eta \text{ real}, \quad \eta > 0.$$

It is the solution of the *resolvent equation*

$$\begin{aligned} (sI - P)\hat{u} &= \hat{F}, \\ L_0\hat{u}(0,s) = \hat{g}_0, &\qquad L_1\hat{u}(1,s) = \hat{g}_1. \end{aligned} \tag{10.3.2}$$

Typical estimates for the solutions of Eqs. (10.3.2) are listed below [see Eqs. (10.1.10) and (10.2.9)]:

1. Consider Eqs. (10.3.2) with homogeneous boundary conditions ($\hat{g}_j \equiv 0$). There is a constant η_0 and a function $K(\eta)$, with $\lim_{\eta\to\infty} K(\eta) = 0$, such that, for all $\hat{F}$ and all s with $\eta = \operatorname{Re} s > \eta_0$,

$$\|\hat{u}(\cdot,s)\|^2 + \delta\|\hat{u}_x(\cdot,s)\|^2 \le K(\eta)\|\hat{F}(\cdot,s)\|^2. \tag{10.3.3a}$$

2. Instead of Eq. (10.3.3a), for inhomogeneous boundary conditions we have

$$\|\hat{u}(\cdot,s)\|^2 + \delta\|\hat{u}_x(\cdot,s)\|^2 \le K(\eta)(\|\hat{F}(\cdot,s)\|^2 + |\hat{g}_0(s)|^2 + |\hat{g}_1(s)|^2). \quad (10.3.4a)$$

(This estimate is stronger than the one derived for the parabolic problem in Section 10.2.)

If the differential operator P is defined on the function space satisfying the homogeneous boundary conditions $L_0v(0) = L_1v(1) = 0$, the operator $(sI-P)^{-1}$ is called the *resolvent operator*, and the *resolvent condition* is usually formulated as

$$\|(sI - P)^{-1}\| \le \frac{\text{constant}}{\operatorname{Re} s}.$$

Our conditions (10.3.3a) and (10.3.4a) are generalized forms of the resolvent condition.

By Parseval's relation, these inequalities imply the following estimates:

$$\int_0^\infty e^{-2\eta t}(\|u(\cdot,t)\|^2 + \delta\|u_x(\cdot,t)\|^2)\,dt$$
$$\le K(\eta)\int_0^\infty e^{-2\eta t}\|F(\cdot,t)\|^2\,dt, \qquad \eta > \eta_0, \quad \lim_{\eta\to\infty} K(\eta) = 0, \quad (10.3.3b)$$

$$\int_0^\infty e^{-2\eta t}(\|u(\cdot,t)\|^2 + \delta\|u_x(\cdot,t)\|^2)\,dt$$
$$\le K(\eta)\int_0^\infty e^{-2\eta t}(\|F(\cdot,t)\|^2 + |g_0(t)|^2 + |g_1(t)|^2)\,dt,$$
$$\eta > \eta_0, \qquad \lim_{\eta\to\infty} K(\eta) = 0, \quad (10.3.4b)$$

respectively.

For the examples treated above we have $\eta_0 = 0$. However, if the problem has variable coefficients, lower order terms, or two boundaries, then we have to choose $\eta_0 > 0$. The last two of these generalizations will be discussed later.

Because the integrals are taken over an infinite time interval, the constant $K(\eta)$ necessarily goes to infinity at some point $\eta = \eta_0$. For example, if $F = g_0 = g_1 = 0$ for $t \ge T$, then the integrals on the right-hand side of the estimates are finite for any value of y. However, the solution u is in general nonzero for

all t, and the integrals on the left-hand side of the estimates do not exist for $y \le y_0$. Therefore, $K(\eta) \to \infty$ as $\eta \to y_0$.

We now use these estimates to introduce a new concept of well-posedness for the systems (10.3.1) with variable coefficients.

Definition 10.3.1. *Consider the problem (10.3.1) with $f \equiv g_0 \equiv g_1 \equiv 0$. We call the problem well posed in the generalized sense if, for any smooth compatible F, there is a smooth solution that satisfies the estimate (10.3.3b) for all $\eta > \eta_0$. Here*

$$\delta = 0 \quad \textit{for hyperbolic problems,}$$
$$\delta > 0 \quad \textit{for parabolic problems,}$$

and η_0, $K(\eta)$ are constants that do not depend on F. We call the problem strongly well posed in the generalized sense if the estimate (10.3.4b) holds.

REMARK. The initial and boundary conditions can be made homogeneous by subtracting a suitable function $\psi(x,t)$ that satisfies Eqs. (10.3.1b) and (10.3.1c). The function $v = u - \psi$ satisfies

$$\partial v/\partial t = Pv + \tilde{F}, \qquad 0 \le x \le 1, \quad t \ge 0,$$
$$v(x,0) = 0,$$
$$L_0v(0,t) = 0, \qquad L_1v(1,t) = 0,$$

where $\tilde{F} = P\psi - \partial\psi/\partial t + F$. Assuming that the problem is well posed in the generalized sense, we obtain the estimate (10.3.3b) as $u \to v$, $F \to \tilde{F}$. Hence, we get an estimate for $u = v + \psi$, but the bound depends on dg_0/dt, dg_1/dt, df/dx, and also on d^2f/dx^2 in the parabolic case.

There are many other ways to define well-posedness. Any definition of well-posedness must satisfy the following requirements.

1. For smooth compatible data, the problem has a smooth solution.
2. There is an estimate of the solution in terms of the data.
3. It should be stable against perturbations of lower order terms; that is, if we perturb the differential equations by changing the lower order terms, then the solutions of the perturbed problem should also satisfy an estimate of the same type.
4. It should hold for a large class of problems.
5. There should be equivalent algebraic conditions that are easy to verify.

mates can be verified for symmetric first-order systems and for parabolic systems by the use of integration by parts. However, there are large classes of problems that cannot be investigated in this way. By using Definition 10.3.1, we can cover a much wider class of problems.

The concept of strong well-posedness in the generalized sense does not play the same role for parabolic equations as for hyperbolic equations. It holds for parabolic equations only if all boundary conditions are derivative conditions, that is, rank$(R_{11}) = m$ in Eq. (10.2.11c).

We now derive a number of fundamental properties for our new concepts.

Theorem 10.3.1. *Assume that the differential operator P is semibounded with the boundary conditions (10.3.1c) such that*

$$\operatorname{Re}(v, Pv) \le -\delta \|v_x\|^2 + \alpha \|v\|^2,$$

where $\delta > 0$ if P is parabolic and $\delta = 0$ if P is hyperbolic. Then, the problem (10.3.1) is well posed in the generalized sense.

Proof. Introduce into Eq. (10.3.1) a new variable, $w = e^{-\eta t}u$. Then, we obtain

$$\partial w/\partial t = (P - \eta I)w + e^{-\eta t}F.$$

Therefore, for $\tilde{\eta} = \eta - \alpha \ge \tilde{\eta}_0 > 0$,

$$\begin{aligned}\frac{d}{dt}\|w\|^2 &\le -2\delta\|w_x\|^2 - 2(\eta - \alpha)\|w\|^2 + 2|(w, e^{-\eta t}F)|,\\ &\le -2\delta\|w_x\|^2 - \tilde{\eta}\|w\|^2 + \frac{1}{\tilde{\eta}}\|e^{-\eta t}F\|^2.\end{aligned}$$

Thus,

$$\|w(\cdot,t)\|^2 \le \int_0^t e^{-\tilde{\eta}(t-\tau)}(G(\tau) - H(\tau))\tau,$$

where

$$G(\tau) = \frac{1}{\tilde{\eta}}\|e^{-\eta\tau}F(\cdot,\tau)\|^2, \qquad H(\tau) = 2\delta\|w_x(\cdot,\tau)\|^2.$$

$$\varphi(t) = \begin{cases} e^{-\tilde{\eta}t}, & \text{for } t \ge 0,\\ 0, & \text{for } t < 0.\end{cases}$$

Then, we obtain

$$\begin{aligned}\int_0^\infty \|w(\cdot,t)\|^2\,dt &\le \int_0^\infty \left(\int_0^\infty \varphi(t-\tau)\,dt\right) G(\tau)d\tau\\ &\quad - \int_0^\infty \left(\int_0^\infty \varphi(t-\tau)\,dt\right) H(\tau)\,d\tau\\ &\le \frac{1}{\tilde{\eta}^2}\int_0^\infty e^{-2\eta t}\|F(\cdot,t)\|^2\,dt - \frac{2\delta}{\tilde{\eta}}\int_0^\infty \|w_x(\cdot,t)\|^2\,dt;\end{aligned}$$

that is,

$$\tilde{\eta}\int_0^\infty \|w(\cdot,t)\|^2\,dt + 2\delta\int_0^\infty \|w_x(\cdot,t)\|^2\,dt \le \frac{1}{\tilde{\eta}}\int_0^\infty e^{-2\eta t}\|F(\cdot,t)\|^2\,dt.$$

Thus, Eq. (10.3.3b) is satisfied with

$$K(\eta) = \max\left(\frac{1}{(y-\alpha)^2}, \frac{1}{y-\alpha}\right)$$

and with δ replaced by $2\delta/\tilde{\eta}_0$.

The existence of a smooth solution can also be verified and, therefore, the theorem is proved.

Now we will prove that lower order terms do not affect generalized well-posedness.

Theorem 10.3.2. *Assume that the problem (10.3.1) is well posed in the generalized sense. Then the perturbed problem*

$$\begin{aligned}\partial w/dt &= (P + P_0)w + F, \qquad 0 \le x \le 1, \qquad t \ge 0,\\ w(x,0) &= 0,\\ L_0 w(0,t) &= 0, \qquad L_1 w(1,t) = 0,\end{aligned}$$

Proof. Formally $P_0 w$ can be considered as a forcing function, and by assumption we have

$$\int_0^\infty e^{-2\eta t}(\|w(\cdot,t)\|^2 + \delta\|w_x(\cdot,t)\|^2)\,dt \le K(\eta)\int_0^\infty e^{-2\eta t}(\|P_0 w(\cdot,t)\|^2 + \|F(\cdot,t)\|^2)\,dt.$$

By choosing η sufficiently large and, therefore, $K(\eta)$ sufficiently small and recalling that $\delta > 0$ for parabolic problems, we can move the $P_0 w$ term to the left-hand side giving us

$$\int_0^\infty e^{-2\eta t}(\|w(\cdot,t)\|^2 + \delta\|w_x(\cdot,t)\|^2)\,dt \le K_1(\eta)\int_0^\infty e^{-2\eta t}\|F(\cdot,t)\|^2\,dt, \qquad \eta > \eta_1, \quad \lim_{\eta\to\infty} K_1(\eta) = 0.$$

The existence of a solution can also be assured and the theorem is proved.

As we will see later, it is convenient to treat each boundary by itself. For that purpose, we define the two quarter-space problems

$$\begin{aligned} \partial u/\partial t &= Pu + F, \qquad 0 \le x < \infty, \quad t \ge 0,\\ u(x,0) &= 0,\\ L_0 u(0,t) &= 0, \end{aligned} \tag{10.3.5}$$

$$\begin{aligned} \partial u/\partial t &= Pu + F, \qquad -\infty < x \le 1, \quad t \ge 0,\\ u(x,0) &= 0,\\ L_1 u(1,t) &= 0, \end{aligned} \tag{10.3.6}$$

and the Cauchy problem

$$\begin{aligned} \partial u/\partial t &= Pu + F, \qquad -\infty < x < \infty, \quad t \ge 0,\\ u(x,0) &= 0. \end{aligned} \tag{10.3.7}$$

The coefficient matrices in P are extended in a smooth way to the whole x axis so that they are constant for large $|x|$. We assume that the functions F have compact support in all three cases.

Definition 10.3.1 of well-posedness in the generalized sense is now the same for each one of these three problems, but with the norms defined by

$$\begin{aligned} \|u(\cdot,t)\|^2 &= \|u(\cdot,t)\|_{0,\infty}^2 := \int_0^\infty |u(x,t)|^2 dx, && \text{for Eq. (10.3.5)},\\ &= \|u(\cdot,t)\|_{-\infty,1}^2 := \int_{-\infty}^1 |u(x,t)|^2 dx, && \text{for Eq. (10.3.6)},\\ &= \|u(\cdot,t)\|_{-\infty,\infty}^2 := \int_{-\infty}^\infty |u(x,t)|^2 dx, && \text{for Eq. (10.3.7)}. \end{aligned}$$

We can now prove the following theorem.

Theorem 10.3.3. *The problem (10.3.1) is well posed in the generalized sense if the quarter-space problems (10.3.5) and (10.3.6) and the Cauchy problem (10.3.7) are all well posed in the generalized sense.*

Proof. Let $\varphi_1(x) \in C^\infty(-\infty,\infty)$ be a monotone function with

$$\varphi_1(x) = \begin{cases} 1, & \text{for } x \le 1/8,\\ 0, & \text{for } x \ge 1/4, \end{cases}$$

and define

$$\begin{aligned} \varphi_2(x) &= \varphi_1(1-x),\\ \varphi_3(x) &= 1 - \varphi_1(x) - \varphi_2(x). \end{aligned}$$

Let

$$\begin{aligned} u_j(x,t) &= \varphi_j(x)u(x,t), \qquad j = 1,2,3,\\ F_j(x,t) &= \varphi_j(x)F(x,t), \qquad j = 1,2,3, \end{aligned}$$

and define $u_1 \equiv 0$ for $x \ge 1$, $u_2 \equiv 0$ for $x \le 0$, and $u_3 = 0$ for $x \le 0$, $x \ge 1$, and, correspondingly, define $F_j(x,t)$. Multiplying Eqs. (10.3.1) by φ_j where $j = 1,2,3$, the problem with homogeneous initial and boundary conditions can be written in the form

$$\begin{aligned} (u_1)_t &= Pu_1 + P_1 u + F_1, \qquad 0 \le x < \infty, \quad t \ge 0,\\ u_1(x,0) &= 0,\\ L_0 u_1(0,t) &= 0, \end{aligned} \tag{10.3.8a}$$

$$\begin{aligned} (u_2)_t &= Pu_2 + P_2 u + F_2, \qquad -\infty < x \le 1, \quad t \ge 0,\\ u_2(x,0) &= 0,\\ L_1 u_2(1,t) &= 0, \end{aligned} \tag{10.3.8b}$$

$$(u_3)_t = Pu_3 + P_3 u + F_3, \qquad -\infty < x < \infty, \quad t \geq 0,$$
$$u_3(x,0) = 0. \tag{10.3.8c}$$

Here $\{P_j\}_1^3$ are bounded matrices in the hyperbolic case. For parabolic equations, $\{P_j u\}_1^3$ are linear combinations of u and its first derivatives.

Now assume that the two quarter-space problems and the Cauchy problem are well posed in the generalized sense. The solutions of Eq. (10.3.8) satisfy

$$\int_0^\infty e^{-2\eta t}(\|u_1\|_{0,\infty}^2 + \delta\|u_{1x}\|_{0,\infty}^2)\,dt \leq K_1(\eta)\int_0^\infty e^{-2\eta t}(\|F_1\|_{0,1}^2 + \|u\|_{0,1}^2 + \delta\|u_x\|_{0,1}^2)\,dt, \quad \eta > \eta_1, \tag{10.3.9a}$$

$$\int_0^\infty e^{-2\eta t}(\|u_2\|_{-\infty,1}^2 + \delta\|u_{2x}\|_{-\infty,1}^2)\,dt \leq K_2(\eta)\int_0^\infty e^{-2\eta t}(\|F_2\|_{0,1}^2 + \|u\|_{0,1}^2 + \delta\|u_x\|_{0,1}^2)\,dt, \quad \eta > \eta_2, \tag{10.3.9b}$$

$$\int_{-\infty}^\infty e^{-2\eta t}(\|u_3\|_{-\infty,\infty}^2 + \delta\|u_{3x}\|_{-\infty,\infty}^2)\,dt \leq K_3(\eta)\int_{-\infty}^\infty e^{-2\eta t}(\|F_3\|_{0,1}^2 + \|u\|_{0,1}^2 + \delta\|u_x\|_{0,1}^2)\,dt, \quad \eta > \eta_3, \tag{10.3.9c}$$

Here we have used the fact that the functions u_j were smoothly extended and the coefficients of P_j vanish outside the intervals $1/8 \leq x \leq 1/4$, $3/4 \leq x \leq 7/8$. The inequalities are added and η is chosen large enough so that the $\|u\|_{0,1}^2$ and $\|u_x\|_{0,1}^2$ terms can be moved to the left-hand side. Observing that $u = u_1+u_2+u_3$ for $0 \leq x \leq 1$, we obtain

$$\begin{aligned}\int_0^\infty e^{-2\eta t}(\|u\|_{0,1}^2 + \delta\|u_x\|_{0,1}^2)\,dt &\leq \text{constant}\int_0^\infty e^{-2\eta t}\sum_{j=1}^3(\|u_j\|_{0,1}^2 + \delta\|u_{jx}\|_{0,1}^2)\,dt \\ &\leq K_4(\eta)\int_0^\infty e^{-2\eta t}\sum_{j=1}^3\|F_j\|_{0,1}^2\,dt \\ &\leq K_5(\eta)\int_0^\infty e^{-2\eta t}\|F\|_{0,1}^2\,dt, \qquad \eta > \eta_4,\end{aligned}$$

where $\lim_{\eta\to\infty} K_5(\eta) = 0$.

One can also use the above representation to prove the existence of solutions of the original problem. By construction, $u = u_1 + u_2 + u_3$ is a solution of the original problem for $x \in [0,1]$. This proves the theorem.

There are no difficulties in generalizing Definition 10.3.1 to several space dimensions. As in Section 9.6, we consider the differential equation in some domain Ω bounded by a smooth curve $\partial\Omega$. The norms $\|\cdot\|$, $|\cdot|$ in Eq. (10.3.3) now represent the L_2 norm over Ω and $\partial\Omega$, respectively. As we show later in Section 10.6, we can split such a problem into a Cauchy problem and a quarter-space problem.

10.4. SYSTEMS WITH CONSTANT COEFFICIENTS IN ONE SPACE DIMENSION

In this section, we consider systems (10.3.1) with constant coefficients. We derive algebraic conditions such that the initial boundary value problem is well posed in the generalized sense.

As in Sections 10.1 and 10.2, we can derive a necessary condition for well-posedness.

Theorem 10.4.1. *The problem is not well posed if the eigenvalue problem*

$$(P - sI)\varphi = 0,$$
$$L_0\varphi(0) = L_1\varphi(1) = 0, \tag{10.4.1}$$

has a sequence of eigenvalues s_j, $j = 1, 2, \ldots,$ with

$$\lim_{j\to\infty} \operatorname{Re} s_j = \infty. \tag{10.4.2}$$

Proof. Assume that there is such a sequence. Denote the corresponding eigenfunctions by $\varphi_j(x)$ with $\|\varphi_j(\cdot)\| = 1$. Then

$$u_j(x,t) = e^{s_j t}\varphi_j(x) \tag{10.4.3}$$

are solutions of Eq. (10.3.1) where

$$u_j(x,0) = \varphi_j(x), \qquad g_0 \equiv g_1 \equiv F \equiv 0.$$

Corresponding to Lemma 10.1.1, the relation

$$\frac{\|u_j(\cdot,t)\|}{\|u_j(\cdot,0)\|} = e^{\operatorname{Re} s_j t}$$

tells us that the problem cannot be well posed. This proves the theorem.

the problem is well posed in the generalized sense.

On the other hand, the boundary condition (10.4.13c) gives us

$$|\hat{u}(0,s)| = |s|^2|\hat{w}(0,s)| \le \frac{|s|^2}{\eta} \|\hat{G}\|^2,$$

that is,

$$\frac{\eta}{2} \|\hat{u}\|^2 \le \frac{1}{2\eta} \|\hat{F}\|^2 + \frac{|s|^2}{2\eta} \|\hat{G}\|^2. \qquad (10.4.14)$$

One can prove that this estimate is sharp and, therefore, that the estimate (10.3.3a) does not hold because $\lim_{\eta\to\infty} K(\eta) \ne 0$. One might think that the definition of well-posedness could be changed such that the last case would be included. However, consider the strip problem

$$\begin{aligned} u_t + u_x &= F, \qquad 0 \le x \le 1, \quad t \ge 0, \\ w_t - w_x &= G, \\ u(x,0) &= w(x,0) = 0, \end{aligned}$$

with boundary conditions

$$u(0,t) = w_t(0,t), \qquad w(1,t) = u(1,t).$$

The corresponding eigenvalue problem is

$$\begin{aligned} s\varphi + \varphi_x &= 0, \\ s\psi - \psi_x &= 0, \\ \varphi(0) &= s\psi(0), \qquad \psi(1) = \varphi(1), \end{aligned}$$

that is,

$$\varphi = e^{-sx}\varphi(0), \qquad \psi = e^{s(x-1)}\psi(1),$$

where

$$\begin{bmatrix} 1 & -se^{-s} \\ e^{-s} & -1 \end{bmatrix} \begin{bmatrix} \varphi(0) \\ \psi(1) \end{bmatrix} = 0.$$

$$s = e^{2s}. \qquad (10.4.15)$$

This equation has solutions with arbitrarily large Re s (see Exercise 10.4.2), and, therefore, the strip problem is not well posed in any computationally suitable meaning.

One can geometrically explain what happens. The characteristics that support w leave the strip at the boundary $x = 0$. To obtain $u(0,t)$, we have to differentiate w; that is, we lose one derivative. The value u is transported to the other boundary, and there it is transferred to w through the boundary condition. This value is again transported to the boundary $x = 0$ and loses another derivative when its value is transferred to u. Thus, we lose more and more derivatives as time increases.

EXERCISES

10.4.1. Prove that the estimate (10.4.14) is sharp, that is, that Eq. (10.3.3a) does not hold.

10.4.2. Prove that Eq. (10.4.15) has solutions s with arbitrarily large Re s.

10.4.3. Prove by direct calculation that the eigenvalues s of

$$\left.\begin{aligned} s\varphi + \varphi_x &= 0, \\ s\psi - \psi_x &= 0, \end{aligned}\right\} \quad 0 \le x \le 1,$$
$$\begin{aligned} s\varphi(0) &= \psi(0), \\ \psi(1) &= \varphi(1), \end{aligned}$$

satisfy Re $s \le \eta_0$ = constant in agreement with the generalized well-posedness of Eqs. (10.4.12a) and (10.4.12b).

10.5. HYPERBOLIC SYSTEMS WITH CONSTANT COEFFICIENTS IN SEVERAL SPACE DIMENSIONS

In this section, we consider hyperbolic systems

$$u_t = Au_x + Bu_y + F =: P\left(\frac{\partial}{\partial x}\right) u + F \qquad (10.5.1a)$$

in the quarter space $x \ge 0$, $-\infty < y < \infty$, $t \ge 0$. For $t = 0$, we give initial data

$$u(\mathbf{x},0) = f(\mathbf{x}), \qquad \mathbf{x} = (x,y) \tag{10.5.1b}$$

and, at $x = 0$, we prescribe boundary conditions

$$\begin{aligned} L_0 u(0,y,t) &= g(y,t), \qquad -\infty < y < \infty, \\ \|u(\cdot,t)\| &< \infty, \end{aligned} \tag{10.5.1c}$$

which are of the same form as in Eq. (10.1.1c).

We assume that all the coefficients are real and A is nonsingular. We also assume that F and g are 2π-periodic in y, and we consider solutions that are 2π-periodic in y.

We want to derive algebraic conditions guaranteeing that the above problem is well posed or strongly well posed in the generalized sense. Corresponding to Eqs. (10.3.3) and (10.3.4), the estimates are now defined as integrals over the domain $0 \le x < \infty$, $0 \le y \le 2\pi$. Corresponding to Lemma 10.1.1, we now have the following lemma.

Lemma 10.5.1. *Consider Eq. (10.5.1) with $F \equiv g \equiv 0$. The problem is not well posed if we can find a complex number s with $\operatorname{Re} s > 0$, an integer ω, and initial data*

$$u(\mathbf{x},0) = e^{i\omega y}\varphi(x), \qquad \|\varphi(\cdot)\| < \infty,$$

such that

$$u(\mathbf{x},t) = e^{st+i\omega y}\varphi(x) \tag{10.5.2}$$

is a solution of Eq. (10.5.1) (the Lopatinsky condition).

Proof. If Eq. (10.5.2) is a solution, so is

$$u_n(\mathbf{x},t) = e^{snt+i\omega ny}\varphi(nx),$$

for any positive integer n. Therefore, we obtain solutions that grow arbitrarily fast and the problem is not well posed.

We now give conditions such that solutions of the form of Eq. (10.5.2) exist. Substituting Eq. (10.5.2) into Eq. (10.5.1) gives us

$$s\varphi = A\varphi_x + i\omega B\varphi, \qquad 0 \le x < \infty, \tag{10.5.3a}$$

$$L_0\varphi(0) = 0, \qquad \|\varphi(\cdot)\| < \infty. \tag{10.5.3b}$$

As before, we have the lemma.

Lemma 10.5.2. *There is a solution of the form of Eq. (10.5.2) if, and only if, for some fixed ω, the eigenvalue problem (10.5.3) has an eigenvalue s with* $\operatorname{Re} s > 0$.

REMARK. ω need not be an integer, because, if we have a solution for s, ω, then we also have a solution for $s/|\omega|$, $\omega/|\omega| = \pm 1$.

By assumption, A is nonsingular, and we can write Eq. (10.5.3a) in the form

$$\varphi_x = M\varphi, \qquad M = A^{-1}(sI - i\omega B).$$

We need the following lemma.

Lemma 10.5.3. *Assume that the system (10.5.1) is strongly hyperbolic. Then there is a constant $\delta > 0$ such that, for $\operatorname{Re} s > 0$, the eigenvalues κ of the matrix M satisfy the estimate*

$$|\operatorname{Re}\kappa| \ge \delta|\operatorname{Re} s|. \tag{10.5.4}$$

Proof. Let β be a real number, and consider

$$\begin{aligned} (M - i\beta I)^{-1} &= (A^{-1}(sI - i\omega B) - i\beta I)^{-1}, \\ &= (sI - i\omega B - i\beta A)^{-1}A. \end{aligned}$$

By assumption, the system is strongly hyperbolic, and, therefore, there is a transformation $T = T(\omega,\beta)$ with $\sup_{\omega,\beta}(|T|\,|T^{-1}|) < \infty$ such that

$$T^{-1}(\omega B + \beta A)T = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_m \end{bmatrix} =: \Lambda, \qquad \lambda_j \text{ real}.$$

Thus,

$$(sI - i\omega B - i\beta A)^{-1}A = T(sI - i\Lambda)^{-1}T^{-1}A;$$

that is,

$$\begin{aligned} |(M - i\beta I)^{-1}| &\le |T|\,|T^{-1}|\,|A|\cdot|(sI - i\Lambda)^{-1}|, \\ &\le \delta|\operatorname{Re} s|^{-1}, \qquad \delta = |A|\,|T^{-1}|\,|T|, \end{aligned} \tag{10.5.5}$$

which implies $|\kappa - i\beta| \ge \delta |\mathrm{Re}\, s|$. Because β is arbitrary, we choose $\beta = \mathrm{Im}\, \kappa$, and Eq. (10.5.4) follows.

The last lemma gives us the following lemma.

Lemma 10.5.4. *Assume that the system (10.5.1a) is strongly hyperbolic. For* $\mathrm{Re}\, s > 0$, *the matrix* M *has no eigenvalues* κ *with* $\mathrm{Re}\, \kappa = 0$. *If* A *has exactly* $m - r$ *negative eigenvalues, then* M *has exactly* $m - r$ *eigenvalues* κ *with* $\mathrm{Re}\, \kappa < 0$, *for all* s *with* $\mathrm{Re}\, s > 0$ *and all real* ω.

Proof. The first statement of the lemma is a weaker statement than Eq. (10.5.4). The eigenvalues κ of M are continuous functions of ω. Therefore, the number of κ with $\mathrm{Re}\, \kappa < 0$ does not depend on ω since $\mathrm{Re}\, \kappa$ cannot change sign if we vary ω. In particular, for $\omega = 0$, we obtain

$$M = sA^{-1},$$

and the second statement of the lemma follows.

Assume for a moment that the eigenvalues of M are distinct and denote by $\kappa_1, \ldots, \kappa_{m-r}$ the eigenvalues with $\mathrm{Re}\, \kappa < 0$. Then, the general solution of Eq. (10.5.3a), belonging to L_2, can be written in the form

$$\varphi = \sum_{j=1}^{m-r} \sigma_j y_j e^{\kappa_j x}.$$

Here the y_j are eigenvectors satisfying

$$My_j = \kappa_j y_j.$$

Substituting this expression into the boundary conditions gives us a linear system of equations for $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_{m-r})$, which we write in the form

$$C(s, \omega)\boldsymbol{\sigma} = 0. \tag{10.5.6}$$

There is a solution of the form of Eq. (10.5.2) if Eq. (10.5.6) has a nontrivial solution.

If the eigenvalues of M are not distinct, then we can still write the general solution, belonging to L_2, in the form

$$\varphi = \sum_j \varphi_j(x) e^{\kappa_j x}, \tag{10.5.7}$$

where now $\varphi_j(x)$ are polynomials in x with vector coefficients containing altogether $m - r$ parameters σ_j. Therefore, we also obtain a linear system of type (10.5.6) in this case.

We have shown the following theorem to be true.

Theorem 10.5.1. *The initial-boundary-value problem (10.5.1) is not well posed if, for some* s *with* $\mathrm{Re}\, s > 0$ *and some* ω,

$$\mathrm{Det}\,(C(s, \omega)) = 0.$$

Now assume that the eigenvalue problem (10.5.3) has no eigenvalue s with $\mathrm{Re}\, s > 0$. We want to show that we can solve the initial-boundary-value problem using Fourier and Laplace transforms. As before, we assume that the initial data are zero. By assumption, the data are 2π-periodic in y, that is, we can expand them into Fourier series with respect to y. For example,

$$F(\mathbf{x}, t) = \sum_{\omega=-\infty}^{\infty} \tilde{F}(x, \omega, t) e^{i\omega y}.$$

Therefore, we can also expand the solution into a Fourier series

$$u(\mathbf{x}, t) = \sum_{\omega=-\infty}^{\infty} \tilde{u}(x, \omega, t) e^{i\omega y}.$$

Substituting this expression into Eq. (10.5.1) gives us, for every frequency ω, a one-dimensional problem

$$\begin{aligned} \tilde{u}_t &= A\tilde{u}_x + i\omega B\tilde{u} + \tilde{F}, \\ \tilde{u}(x, \omega, 0) &= 0, \\ L_0\tilde{u}(0, \omega, t) &= \tilde{g}(\omega, t). \end{aligned} \tag{10.5.8}$$

We can solve Eq. (10.5.8) using the Laplace transform. The equation

$$\hat{u}(x, \omega, s) = \int_0^\infty e^{-st}\tilde{u}(x, \omega, t)\, dt$$

satisfies

$$\begin{aligned} s\hat{u} &= A\hat{u}_x + i\omega B\hat{u} + \hat{F}, \qquad \|\hat{u}\| < \infty, \\ L_0\hat{u}(0, \omega, s) &= \hat{g}(\omega, s). \end{aligned} \tag{10.5.9}$$

By assumption, the eigenvalue problem (10.5.3) has no eigenvalue with $\operatorname{Re} s > 0$, and, therefore, we can solve Eq. (10.5.9) for $\operatorname{Re} s > 0$ and every ω. Inverting the Laplace and Fourier transforms gives us the desired solution.

By Parseval's relation, the estimates (10.3.3a) and (10.3.4a) now take the form

$$\|\hat{u}(\cdot,\omega,s)\|^2 \le K(\eta)\|\hat{F}(\cdot,\omega,s)\|^2 \qquad (10.5.10)$$

and

$$\|\hat{u}(\cdot,\omega,s)\|^2 \le K(\eta)\,(\|\hat{F}(\cdot,\omega,s)\|^2 + |\hat{g}(\omega,s)|^2), \qquad ((10.5.11)$$

respectively. Here $K(\eta)$ does not depend on ω.

We now consider the case where $\hat{F} \equiv 0$ and write the differential equation (10.5.9) in the form

$$\begin{aligned} \hat{u}_x &= \tau A^{-1}(s' - i\omega' B)\hat{u} =: \tau M\hat{u}, \\ L_0\hat{u}(0,\omega,s) &= \hat{g}(\omega,s), \end{aligned} \qquad (10.5.12)$$

where

$$\tau = \sqrt{|s|^2 + \omega^2}, \qquad s' = \frac{s}{\tau}, \quad \omega' = \frac{\omega}{\tau}.$$

By Lemma 10.5.4, for every s', ω' with $\operatorname{Re} s' > 0$, the eigenvalues κ of M split into two groups. By Schur's lemma, we can find a unitary transformation $U = U(\omega', s')$ such that

$$U^*(\omega',s')M(\omega',s')U(\omega',s') = \begin{bmatrix} M_{11} & M_{21} \\ 0 & M_{22} \end{bmatrix},$$

where the eigenvalues κ of M_{11} and M_{22} satisfy $\operatorname{Re}\kappa < 0$ and $\operatorname{Re}\kappa > 0$, respectively. Substituting a new variable $\hat{w} = U^*\hat{u}$ into Eq. (10.5.12), we obtain

$$\begin{aligned} \hat{w}^I_x &= \tau M_{11}\hat{w}^I + \tau M_{12}\hat{w}^{II}, \\ \hat{w}^{II}_x &= \tau M_{22}\hat{w}^{II}, \\ L_0 U\hat{w} &=: C^I(\omega',s')\hat{w}^I(0,\omega,s) + C^{II}(\omega',s')\hat{w}^{II}(0,\omega,s) = \hat{g}. \end{aligned} \qquad (10.5.13)$$

Since we are interested in solutions with $\|w\| < \infty$ and the eigenvalues of M_{22} have a positive real part, it follows that

$$\begin{aligned} \hat{w}^I(x,\omega,s) &= e^{\tau M_{11}x}\hat{w}^I(0,\omega,s), \qquad \hat{w}^{II} \equiv 0, \\ C^I(\omega',s')\hat{w}^I(0,\omega,s) &= \hat{g}. \end{aligned} \qquad (10.5.14)$$

There are two possibilities: Firstly, there exist s'_*, ω'_* with $\operatorname{Re} s'_* \ge 0$ and sequences s'_ν, ω'_ν with $\lim_{\nu\to\infty} s'_\nu = s'_*$, $\lim_{\nu\to\infty}\omega'_\nu = \omega'_*$ such that

$$\lim_{\nu\to\infty} |(C^I(\omega'_\nu, s'_\nu))^{-1}| = \infty. \qquad (10.5.15)$$

One can prove that we can choose U such that it is continuous at ω'_*, s'_*. Therefore, Eq. (10.5.15) holds if, and only if,

$$\operatorname{Det}(C^I(\omega'_*, s'_*)) = 0.$$

If $\operatorname{Re} s'_* > 0$, then the homogeneous equations (10.5.14) have a nontrivial solution and, therefore, $s = \tau s'_*$ are eigenvalues of the eigenvalue problem (10.5.3) for $\omega = \tau\omega'_*$. Thus, the problem is not well posed in any sense. If $\operatorname{Re} s_* = 0$, then we obtain a solution of Eq. (10.5.3a) that satisfies $L_0\varphi = 0$ but might not belong to $L_2(0,\infty)$ because some of the eigenvalues of M_{11} might be purely imaginary [cf. Eq. (10.5.4)]. We make the following definition.

Definition 10.5.1. *If* $\operatorname{Det}(C^I(\omega'_*, s'_*)) = 0$, *where s'_* is purely imaginary, then s_* defined by* $s_* = s'\sqrt{|s_*|^2 + \omega^2}$ *is called a* generalized eigenvalue *of the eigenvalue problem (10.5.3) if* $\|\varphi\| \notin L_2(0,\infty)$.

REMARK. Even if s'_* is purely imaginary, the corresponding eigenfunction φ might belong to $L_2(0,\infty)$, that is, $\operatorname{Re}\kappa_\nu < 0$, $\nu = 1,\ldots,m-r$. In such a case s_* is an eigenvalue.

The theory for the case with generalized eigenvalues, or eigenvalues on the imaginary axis is incomplete. In some cases the initial-boundary-value problem is well posed in the generalized sense, in other cases it is not. We discuss this in more detail for difference approximations.

Secondly, the alternative to Eq. (10.5.15) is that $(C^I(\omega',s'))^{-1}$ is uniformly bounded, or equivalently, that the *determinant condition* is fulfilled:

$$\operatorname{Det}(C^I(\omega',s')) \ne 0, \qquad |\omega'| \le 1, \quad |s'| \le 1, \quad \operatorname{Re} s' \ge 0. \qquad (10.5.16)$$

This is a strengthened version of the Lopatinsky condition given in Lemma 10.5.1. We now have the following lemma.

Lemma 10.5.5. *Consider the initial-boundary-value problem (10.5.1) with $f = F = 0$ and with g satisfying* $\int_0^\infty\int_0^{2\pi} |g(y,t)|^2\,dy\,dt < \infty$. *Then there is a constant*

$K > 0$ such that its solutions satisfy

$$\int_0^\infty \int_0^{2\pi} |u(0,y,t)|^2\, dy\, dt \le K \int_0^\infty \int_0^{2\pi} |g(y,t)|^2\, dy\, dt \qquad (10.5.17)$$

if, and only if, Eq. (10.5.16) holds.

Proof. First assume that Eq. (10.5.16) holds. Then, because $(C^I)^{-1}$ is uniformly bounded, the solution $\hat{w}$ of Eq. (10.5.13) satisfies

$$|\hat{w}^I(0,\omega,s)|^2 \le K|\hat{g}(\omega,s)|^2, \qquad \operatorname{Re} s > 0, \qquad (10.5.18a)$$

where K is a constant independent of ω, s. The vector function $\hat{u} = U\hat{w}$ satisfies Eq. (10.5.9) with $\hat{F} = 0$, and, because U is a unitary matrix we have $|\hat{u}| = |\hat{w}|$. Therefore,

$$|\hat{u}(0,\omega,s)|^2 \le K|\hat{g}(\omega,s)|, \qquad \operatorname{Re} s > 0. \qquad (10.5.18b)$$

By Parseval's relation, this inequality implies

$$\int_0^\infty \int_0^{2\pi} e^{-2\eta t}|u(0,y,t)|^2\, dy\, dt \le K\int_0^\infty \int_0^{2\pi} e^{-2\eta t}|g(y,t)|^2\, dy\, dt,$$
$$\le K\int_0^\infty \int_0^{2\pi} |g(y,t)|^2\, dy\, dt, \qquad \eta > 0.$$

Because the right-hand side is independent of η, Eq. (10.5.17) follows.

Next assume that Eq. (10.5.17) holds. Then, by Parseval's relation, we get the corresponding integral inequality in the Fourier-Laplace space, and, as demonstrated above, this leads to the pointwise estimate (10.5.18b) for arbitrary g. From this estimate, we obtain Eq. (10.5.18a), which is equivalent to Eq. (10.5.16). This proves the lemma.

We now make the following definition.

Definition 10.5.2. *Consider the system (10.5.9) for $\hat{F} = 0$. If its solutions satisfy Eq. (10.5.18b), we say that it satisfies the* Kreiss condition.

Because the constant K is independent of ω', s', one might think that the condition $\operatorname{Re} s > 0$ could be replaced by $\operatorname{Re} s \ge 0$. However, the reason for keeping the strict inequality is that it automatically selects the correct general solution $\hat{u}$ through the condition $\|\hat{u}\| < \infty$, because the exponentially growing part is annihilated.

Using the arguments above, we get the following lemma.

Lemma 10.5.6. *The Kreiss condition is satisfied if, and only if, the eigenvalue problem (10.5.3) has no eigenvalue or generalized eigenvalue for* $\operatorname{Re} s \ge 0$.

The main result of the theory is presented in the following theorem.

Theorem 10.5.2. *Assume that Eq. (10.5.1a) is a strictly hyperbolic system. If the Kreiss condition is satisfied, then the initial boundary value problem is strongly well posed in the generalized sense.*

We will not give a proof here. In applications it is not necessary to go through the transformation process leading to the formulation (10.5.13). The general solution $\hat{u}$ of Eq. (10.5.9) with $\|\hat{u}\| < \infty$ for $\hat{F} = 0$ and $\operatorname{Re} s > 0$ is obtained just as for the eigenvalue problem, and we arrive at a system

$$C(s,\omega)\boldsymbol{\sigma} = \tilde{g}, \qquad \operatorname{Re} s > 0, \qquad (10.5.19)$$

where $C(s,\omega)$ is the matrix occurring in Eq. (10.5.6). With the proper normalization, the Kreiss condition is equivalent to

$$\operatorname{Det}(C(s,\omega)) \ne 0, \qquad \operatorname{Re} s \ge 0. \qquad (10.5.20)$$

Note that $C(s,\omega)$ must always be defined for $\operatorname{Re} s = 0$ as a limit when s is approaching the imaginary axis from the right. We demonstrate the procedure in an example at the end of this section.

If a hyperbolic problem is well posed, then we have proved that it is also well posed in the generalized sense. One might conjecture that the converse is also true, but no general results are known. However, for strictly hyperbolic equations, we have the following theorem.

Theorem 10.5.3. *Assume that Eq. (10.5.1a) is strictly hyperbolic or symmetric hyperbolic. If the Kreiss condition is satisfied, then the initial-boundary-value problem is strongly well posed.*

Proof. We will only prove this result for symmetric hyperbolic systems. Without restriction, we can assume that

$$A = \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix},$$

$$\Lambda_1 = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_r \end{bmatrix} > 0, \qquad \Lambda_2 = \begin{bmatrix} \lambda_{r+1} & & 0 \\ & \ddots & \\ 0 & & \lambda_m \end{bmatrix} < 0,$$

is diagonal. We first solve an auxiliary problem

$$\begin{aligned} v_t &= Av_x + Bv_y, \\ v(\mathbf{x}, 0) &= f(\mathbf{x}), \\ v^{II}(0, y, t) &= 0, \qquad v^{II} = (v^{(r+1)}, \ldots, v^{(m)})^T, \qquad \|v(\cdot, t)\| < \infty. \end{aligned} \tag{10.5.21}$$

For its solution, we have the energy estimate

$$\frac{d}{dt} \|v\|^2 + \int_0^{2\pi} \langle v(0, y, t), Av(0, y, t)\rangle \, dy = 0;$$

that is,

$$\begin{aligned} &\|v(\cdot, t)\|^2 \le \|v(\cdot, 0)\|^2 = \|f(\cdot)\|^2, \\ &(\min_{1 \le j \le r} \lambda_j) \int_0^T \int_0^{2\pi} |v(0, y, t)|^2 \, dy \, dt < \int_0^T \int_0^{2\pi} \langle v(0, y, t), Av(0, y, t)\rangle \, dy \, dt, \\ &\qquad \le \|f(\cdot)\|^2. \end{aligned} \tag{10.5.22}$$

We assume that $v(x, t)$ is a smooth function of x, t. The difference $w = u - v$ satisfies

$$\begin{aligned} w_t &= Aw_x + Bw_y, \\ w(\mathbf{x}, 0) &= 0, \\ L_0 w(0, y, t) &= g(y, t), \qquad g = -L_0 v(0, y, t), \qquad \|w(\cdot, t)\| < \infty. \end{aligned} \tag{10.5.23}$$

If the Kreiss condition is satisfied, then

$$\begin{aligned} &\int_0^\infty \int_0^{2\pi} |w(0, y, t)|^2 \, dy \, dt \\ &\qquad \le \text{constant} \int_0^\infty \int_0^{2\pi} |g(y, t)|^2 \, dy \, dt \le \text{constant} \|f(\cdot)\|^2. \end{aligned}$$

Thus, we can estimate the solution of Eq. (10.5.23) on the boundary. We can use integration by parts to estimate $\|w(\cdot, t)\|$ and obtain

$$\begin{aligned} \frac{d}{dt} \|w\|^2 &= -\int_0^{2\pi} \langle w(0, y, t), Aw(0, y, t)\rangle \, dy, \\ &\le \text{constant} \int_0^{2\pi} |w(0, y, t)|^2 \, dy; \end{aligned}$$

that is,

$$\|w(\cdot, T)\|^2 \le \text{constant} \int_0^T \int_0^{2\pi} |w(0, y, t)|^2 \, dy \, dt \le \text{constant} \|f(\cdot)\|^2. \tag{10.5.24}$$

The estimates (10.5.22) for v yield the final estimate for $u = v + w$, which shows that for symmetric hyperbolic systems the initial-boundary-value problem is strongly well posed if the Kreiss condition is satisfied.

As an example, we now discuss the system

$$\frac{\partial u}{\partial t} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \frac{\partial u}{\partial x} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \frac{\partial u}{\partial y}, \qquad u = \begin{bmatrix} u^{(1)} \\ u^{(2)} \end{bmatrix}, \tag{10.5.25a}$$

with boundary conditions

$$u^{(1)}(0, y, t) = au^{(2)}(0, y, t) + g, \tag{10.5.25b}$$

where a is a complex constant. Integration by parts gives us

$$\begin{aligned} \frac{d}{dt} \|u\|^2 &= -\int_0^{2\pi} \left\langle u(0, y, t), \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} u(0, y, t)\right\rangle dy, \\ &= \int_0^{2\pi} \left(|u^{(1)}(0, y, t)|^2 - |u^{(2)}(0, y, t)|^2 \right) dy, \\ &= (|a|^2 - 1) \int_0^{2\pi} |u^{(2)}(0, y, t)|^2 \, dy. \end{aligned}$$

Thus, we obtain an energy estimate for $|a| \le 1$.

We want to discuss whether we can also estimate the solution for other values of a using the Laplace transform. The eigenvalue problem (10.5.3) has the form

$$\varphi_x = \begin{bmatrix} -s & i\omega \\ -i\omega & s \end{bmatrix} \varphi =: M\varphi,$$
$$\varphi^{(1)}(0) = a\varphi^{(2)}(0), \qquad \|\varphi\| < \infty.$$

For $\operatorname{Re} s > 0$, M has exactly one eigenvalue

$$\kappa = -\sqrt{s^2+\omega^2}, \qquad \operatorname{Re}\kappa < 0,$$

with negative real part. The corresponding eigenvector is given by

$$\omega^2 \mathbf{e} = (s + \sqrt{s^2+\omega^2},\, i\omega)^T.$$

Therefore,

$$\varphi(x) = \sigma e^{\kappa x}\mathbf{e}.$$

Thus, the problem is not well posed if the relation

$$s + \sqrt{s^2+\omega^2} = ia\omega, \qquad \operatorname{Re} s > 0, \quad \omega \text{ real}, \tag{10.5.26}$$

has a solution. A simple calculation shows that Eq. (10.5.26) has a solution if, and only if, $|a| > 1$, $\operatorname{Im} a \neq 0$. In that case, the problem is not well posed. We have already shown that the problem is well posed if $|a| \leq 1$. Thus, we need only discuss the case $|a| > 1$, where a is real. The problem (10.5.12) has the form

$$\hat{u}_x = \begin{bmatrix} -s & i\omega \\ -i\omega & s \end{bmatrix} \hat{u}, \qquad \hat{u} = \begin{bmatrix} \hat{u}^{(1)} \\ \hat{u}^{(2)} \end{bmatrix},$$
$$\hat{u}^{(1)}(0,\omega,s) = a\hat{u}^{(2)}(0,\omega,s) + \hat{g}(\omega,s), \qquad \|\hat{u}(\cdot,\omega,s)\| < \infty, \tag{10.5.27}$$

where we have kept the original variables s, ω instead of the scaled ones s', ω'. The general solution of the differential equation, belonging to L_2, is given by

$$\hat{u} = \sigma \begin{bmatrix} s + \sqrt{s^2+\omega^2} \\ i\omega \end{bmatrix} e^{-\sqrt{s^2+\omega^2}x}.$$

σ is determined by the boundary condition

$$\sigma(s + \sqrt{s^2+\omega^2} - ia\omega) = \hat{g};$$

that is,

$$\hat{u} = \frac{\hat{g}}{s + \sqrt{s^2+\omega^2} - ia\omega} \begin{bmatrix} s + \sqrt{s^2+\omega^2} \\ i\omega \end{bmatrix} e^{-\sqrt{s^2+\omega^2}x}.$$

We now show that $|\hat{u}(0,\omega,s)|/|\hat{g}(\omega,s)|$ is unbounded. Thus, the Kreiss condition is not satisfied. Choose the sign of ω so that $\omega a = |\omega a|$ and determine $\xi_1 > 1$ from

$$\xi_1 + \sqrt{\xi_1^2 - 1} = |a|.$$

Let $s = i|\omega|\xi_1 + \eta$, $\eta \ll |\omega|$. Then

$$\begin{aligned}
&\lim_{|\omega|\to\infty} (|\hat{u}^{(2)}(0,\omega,s)|/|\hat{g}(\omega,s)|) \\
&= \lim_{|\omega|\to\infty} |i\omega/(s + \sqrt{s^2+\omega^2} - ia\omega)| \\
&= \lim_{|\omega|\to\infty} |\omega/|i|\omega|\xi_1 \\
&\quad + \eta + |\omega|\sqrt{1 - \xi_1^2 + 2i\xi_1\eta/|\omega| + (\eta/\omega)^2} - i|a\omega||| \\
&= \lim_{|\omega|\to\infty} |\omega/|i|\omega|\xi_1 + \eta \\
&\quad + i|\omega|\sqrt{\xi_1^2 - 1}\,(1 + i\xi_1\eta/(|\omega|(1 - \xi_1^2)) \\
&\quad + \mathcal{O}((\eta/\omega)^2)) - i|a\omega||| \\
&= \lim_{|\omega|\to\infty} |\omega/|\eta + i\xi_1\eta/\sqrt{1-\xi_1^2} + \mathcal{O}(\eta^2/\omega)|| = \infty.
\end{aligned}$$

Thus, we cannot obtain the estimate (10.5.18b), and the problem is not strongly well posed. One can show that it is also not well posed in the generalized sense.

In the example above, energy estimates and Laplace transform techniques yield the same restriction for the boundary conditions. Generally, however, Laplace transform techniques give a much wider class of admissible boundary conditions.