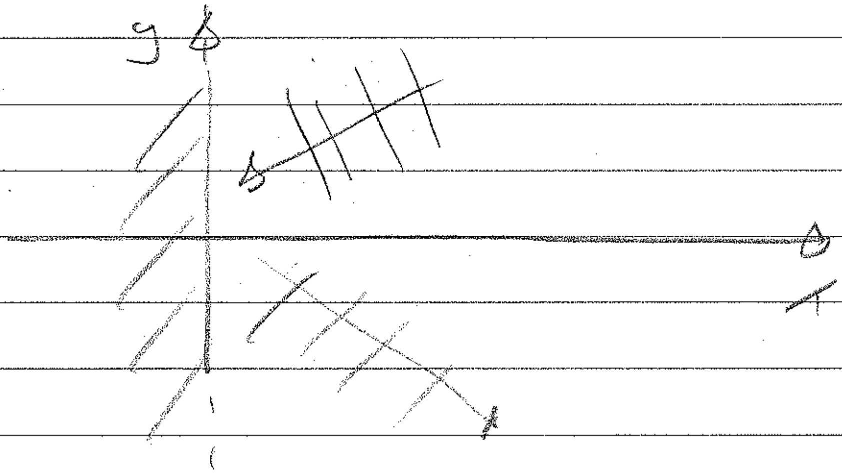
A Short Course in "Boundary Condition Theory": PART I



$$\begin{cases} U_t = AU_x + BU_y & ; \quad U = \begin{pmatrix} V \\ \alpha_x \\ \alpha_y \end{pmatrix} \quad \begin{matrix} U \to 0 \\ y \to \pm\infty \end{matrix} \\ LU = g & x \geq 0 \,,\ -\infty < y < \infty \\ U(x,y,0) = f(x,y) & t \geq 0 \\ \left(A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \right) \end{cases}$$

Energy-method

$$\int_{-\infty}^{\infty}\int_{0}^{\infty} UU_t\,dx\,dy = \int_{-\infty}^{\infty}\int_{0}^{\infty} (UAU)_x\,dx\,dy + \underbrace{\int_{0}^{\infty}\int_{-\infty}^{\infty} \tfrac{1}{2}(UBU)_y\,dx}_{=0}$$

$$\|U\|^2_t = -\int_{-\infty}^{\infty} U^T A U\,dy$$

2)

How many boundary conditions?

Eigenvalues to A?

$$\begin{vmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} \overset{?}{=} -\lambda(\lambda^2-1) = 0 \Rightarrow$$

$\lambda = 0, \quad \lambda = 1, \quad \lambda = -1$

$\therefore$ One negative $\Rightarrow$ 1 B.C. = 1 eq.

$LU = g$ has 1 row.

Note: Not 1×3 b.c's = 3 eq's

components

What conditions?

$$-\int_{-\infty}^{\infty} u^T A u \, dy = -\int_{-\infty}^{\infty} 2u\alpha_x \, dy$$

choose for example:

$v = \alpha\alpha_x + g \Rightarrow L = [1 \; -\alpha \quad 0]$

3)

$$\therefore \frac{d}{dt}\|u\|^2 = -\int_{-\infty}^{\infty} 2d_x(\alpha d_x + q)\,dy$$

Check by putting $q = \text{data} = 0$.

if ok $\therefore$ <u>Well-posed</u>!

Strongly Well-posed? $q \neq 0$

let $\alpha > 0 \Rightarrow 2\alpha d_x^2 + 2d_x q =$

$$= \frac{1}{2\alpha}q^2 + 2\alpha d_x^2 + 2d_x q - \frac{1}{2\alpha}q^2 =$$

$$-\frac{1}{2\alpha}q^2 + \left(\sqrt{2\alpha}\,d_x + \frac{1}{\sqrt{2\alpha}}q\right)^2 \Rightarrow$$

$$\frac{d}{dt}\|u\|^2 = -\int_{-\infty}^{\infty} -\frac{q^2}{2\alpha} + \left(\sqrt{2\alpha}\,d_x + \frac{1}{\sqrt{2\alpha}}q\right)^2 dy \leq$$

$$\frac{1}{2\alpha}\int_{-\infty}^{\infty} q^2\,dy$$

$\therefore$ Strongly Well-posed. for $\alpha > 0$.

4)

Normal Mode Analysis

$u = e^{st + ikx + i\omega y} \Rightarrow$ (u = periodic)

$(sI - A - i\omega B)\psi = 0$, non-triv.

solution $\psi \neq 0 \Rightarrow$

$\det(sI - Ak - i\omega B) = 0 \Rightarrow$

$s(k^2 + \omega^2) = 0$.

— x —

Special case $s = 0$, steady case.

$\Rightarrow \quad (d_x)_x + (d_y)_y = 0$, $\operatorname{div}(d_x, d_y) = 0$

$V_x = 0$

$\Rightarrow V = \text{const.}$

$V_y = 0$

$V = g \ (x = 0)$ seems a good b.c.

5) $s \neq 0 \Rightarrow$

$k_-^2 < 0 \Rightarrow 1$ b.c. at $x=0$

$$k_\pm = \pm\sqrt{s^2+\omega^2} \qquad k_+^2 > 0 \Rightarrow 1 \text{ b.c. at } x=1$$

<u>$k = k_+$</u> $\Rightarrow$

$$\begin{pmatrix} s & -k_+ & -i\omega \\ -k_+ & s & 0 \\ -i\omega & 0 & s \end{pmatrix} \psi_+ = 0$$

$$\Rightarrow \quad \psi_+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \frac{k_+}{s} \\ \frac{i\omega}{s} \end{pmatrix}$$

$$k = k_- \Rightarrow \quad \psi_- = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \frac{k_-}{s} \\ \frac{i\omega}{s} \end{pmatrix}$$

$$\hat{u} = \alpha_+ \psi_+ e^{k_+ x} + \alpha_- \psi_- e^{k_- x}$$

only 2 unknowns

$\Rightarrow$

only 2 b.c.s.

Note only <u>2 totally</u> and

if 1 boundary only <u>one</u>

6)

We consider one boundary at

$x = 0 \Rightarrow \alpha_+ = 0$

$\hat{u} = \alpha_- \psi_- e^{k_- x}$

$\hat{v} - a \hat{\alpha}_x = \hat{g} \Rightarrow$

$\alpha_- \psi_-^{(1)} - a \alpha_- \psi_-^{(2)} = \hat{g} \Rightarrow$

$\underbrace{\left(\psi_-^{(1)} - a \psi_-^{(2)}\right)}_{C} \alpha_- = \hat{g}$

$C(s) = \left(1 - a \frac{k_-}{s}\right) = \left(1 + a \frac{\sqrt{s^2 + \omega^2}}{s}\right) =$

$1 + a\sqrt{1 + \left(\frac{|\omega|}{s}\right)^2}$, Note $s \neq 0$.

$\frac{|\omega|}{s} = \frac{|\omega|(\eta - i\xi)}{\eta^2 + \xi^2} = \frac{|\omega|\eta}{\eta^2+\xi^2} - \frac{i\xi|\omega|}{\eta^2+\xi^2} = \tilde{\eta} + i\tilde{\xi}$

$\therefore \; C(\hat{s}) = 1 + a\sqrt{1 + \tilde{s}^2}$

7)

Obviously for Real $s > 0$ also Real $(\hat{s}) > 0$.

$C(\hat{s}) \neq 0$ for Real $(\hat{s}) > 0$

∴ Lopatinsky cde. no exp. growth.

How about Kreiss cond.?

Let $\hat{s} = i\hat{\xi} \Rightarrow$

$C = 1 + \alpha\sqrt{1 - \hat{\xi}^2}$

1) $\hat{\xi}^2 < 1 \Rightarrow C > 10$

2) $\hat{\xi}^2 = 1 \Rightarrow C = 1 \neq 0$

3) $\hat{\xi}^2 > 1 \Rightarrow C = 1 + \alpha i\sqrt{\hat{\xi}^2 - 1} \neq 0$

No possibility for failure

∴ $|\hat{d}_1| \le |\hat{g}| / |C|$

8)

Note also $\alpha = 0$ ok

since in that case

$$C = 1 \Rightarrow \alpha = \hat{q} \Rightarrow$$

$$\hat{u} = \hat{q} e^{bx} \Rightarrow$$

$$\hat{u}(x, \omega, s) \leq \hat{q}(\omega, s) \quad + \text{Parseval}$$

$$\Rightarrow \int_0^T |u|^2 dt \leq \int_0^T q^2 dt$$

Very Nice result.

9)

A short Course in Weeb Implementation of Boundary Conditions, Part II

$$u_t = Au_x + B\cancel{u_y}$$

$$Lu = g \qquad x \geq 0$$

$$u(x,y,0) = f(x)$$

$$L = [1 \;\; -\alpha \;\; 0] \qquad u = \begin{pmatrix} u \\ u_x \\ u_y \end{pmatrix}$$

We skip the y-dependence for now.

$$\hat{u}_t = (P^{-1}Q \otimes A)\hat{u} + P^{-1}(e_0 \times \tau_0)(Lu - g)$$

$$\tilde{u}(0) = f$$

position in space

to be determined

from PDE

Energy $\Rightarrow$

$$u^T(P \otimes I)u_t = u^T(Q \otimes A)u + u_0^T \tau_0 (Lu - g)$$

size of $A = 3 \times 3$

10)

Add transpose $\Rightarrow$

$$\left(\|\mathcal{U}\|^2_{P_{0L}}\right)_t = \underbrace{U^T\left(A\epsilon Q^T \otimes A\right) + U_0\left(T_0 L + L^T T_0\right) U}_{\text{collect terms at } t=0}$$

$$\underbrace{- U_0^T T_0 q - q^T T_0^T U_0}_{\text{ignore from the start.}}$$

$$BT = U_0^T \underbrace{\left(-A + T_0 L + (T_0 L)^T\right)}_{\hat{A}} U_0 =$$

L = is assumed.

$$\begin{pmatrix} T_1 \\ T_2 \\ T_3 \end{pmatrix} \begin{pmatrix} 1 & -\alpha & 0 \end{pmatrix} = \begin{pmatrix} T_1 & -\alpha T_1 & 0 \\ T_2 & -\alpha T_2 & 0 \\ T_3 & -\alpha T_3 & 0 \end{pmatrix}$$

+ Transpose $\Rightarrow$

$$\hat{A} = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 2T_1 & T_2 - \alpha T_1 & T_3 \\ T_2 - \alpha T_1 & -2\alpha T_2 & -\alpha T_3 \\ T_3 & -\alpha T_3 & 0 \end{pmatrix}$$

$$\le 0 \Rightarrow \quad T_1 = T_3 = 0 \quad T_2 = 1 \quad \Rightarrow$$

$$\hat{A} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2\alpha & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

11)

How about data? Well totally we get

$$BT = -2\alpha d_x^2 - u_0^T T_0\, g - g^T T_0^T u_0 = -2\alpha d_x^2 - 2d_x g$$

$$= \frac{g^2}{2\alpha} - \frac{g^2}{2\alpha} - 2\alpha d_x^2 - 2d_x g =$$

$$\frac{g^2}{2\alpha} - \left(\sqrt{2\alpha}\, d_x + \frac{1}{\sqrt{2\alpha}}\, g\right)^2$$

$\therefore$ exactly as in PDE case.