1) Strict Stability

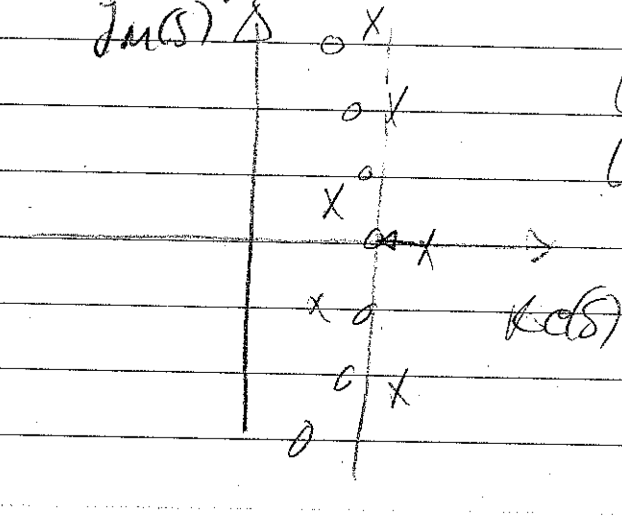
Continuous $\|u\| \le K_c e^{\alpha_c t} \|u(0)\|$ (1)

Semi-discrete $\|u_h\| \le K_h e^{\alpha_h t} \|u(0)_h\|$ (2)

Def: If $\alpha_d \le \alpha_c + O(h)$ (3)
then strictly stable

Note 1 Estimate for Cont. case
must be sharp.

Note 2 Maximum "growth" eigenvalue
must be captured.



(O) = cont
(x) = discrete

2)

Constant Coeff. (prototype 1)

$u_t + a u_x = 0 \; ; \; 0 \le x \le 1$

$u(0,t) = g$

$u(x,0) = f(x)$

Energy $\Rightarrow \|u\|_t^2 = a g^2 - u(1,t)^2$

$g = 0 \Rightarrow \|u\|^2 \le \|f\|^2 \quad K_c = 1 \cdot \alpha_c = 0$

SBP & SAT

$u_t + a P^{-1} Q = P^{-1} \lambda (u_0 - g) e_0$

$u(0) = f$

Energy $\Rightarrow (\|u\|_h^2)_t = u_0^2 (1+2\lambda) - u_N^2 - 2\lambda u_0 g$

$g = 0 \Rightarrow \|u\|_h^2 \le \|f\|_h^2 \quad$ if $1+2\lambda \le 0$

$K_d = K_c = 1$, and $\alpha_d = \alpha_c = 0$

$\therefore$ Strictly Stable

3)

Varible Coeff (prototype 2)

$$u_t + a(x,t)u_x = 0, \quad 0 \le x \le c, \quad t > 0$$
$$u(0,t) = g \qquad a > 0$$
$$u(x,0) = f$$

$$\alpha(au)_x + \beta a_x u + \gamma a u_x = a u_x \Rightarrow$$

$$(\alpha+\beta)a_x u + (\alpha+\gamma) a u_x = a u_x$$

$$\alpha+\beta = 0 \quad \beta = -\alpha \quad \gamma = 1-\alpha$$

$$u_t + \alpha(au)_x - \alpha a_x u + (1-\alpha) a u_x = 0$$

$$\int u u_t + \int \alpha u (au)_x - \alpha a_x u^2 + (1-\alpha) u a u_x \, dx = 0$$

$$\frac{1}{2}\|u\|^2 + \int \alpha a u^2 - \alpha a u u_x + (1-\alpha) a u u_x \, dx$$

$$= \alpha \int a_x u^2 \, dx$$

4)

choose $-\alpha + 1 - \alpha = 0 \Rightarrow \alpha = 1/2$

$$\|u\|_t^2 = au\big|_0 - au\big|_1 + \int_0^1 a u\, dx =$$

$$= a(0,t)g^2 - a(1,t)u^2(1,t) + \int_0^1 a_x u^2\, dx \quad (*)$$

<u>Semidiscrete</u>

$$u_t + \tfrac{1}{2} P^{-1} Q (Au) - \tfrac{1}{2} A_x u + \tfrac{1}{2} A P^{-1} Q u =$$
$$P^{-1} \alpha (u_0 - g) e_0$$

$$u(0) = f$$

Energy $\Rightarrow$

$$\cancel{u^T P u_t + \underbrace{\tfrac{1}{2} u^T Q A u + \tfrac{1}{2} u^T A Q u}_{\tfrac{1}{2} u^T B A u \, - \, \underbrace{\tfrac{1}{2} (Qu)^T A u + \tfrac{1}{2} (Au)^T Q u}_{=0}} = u^T P A_x u}$$

5)

$$\Rightarrow \frac{d}{dt}\|u\|^2 = a_0(t)u_0^2 - a_N(t)u_N^2 + \underbrace{u^T P A u}_{\approx \int_0^1 a u_x^2 dx}$$

$+ 2u_0\alpha(u_0 - g)$ (**)

$BT\ u_0^2(a_0 + 2\alpha) - a_N(t)u_N^2 - 2\alpha u_0 g$

Let $\alpha = -a_0 \Rightarrow BT = a_0 g^2 - a_N u_N^2 - a_0(u_0 - g)^2$

For comparision of (*) and (**)

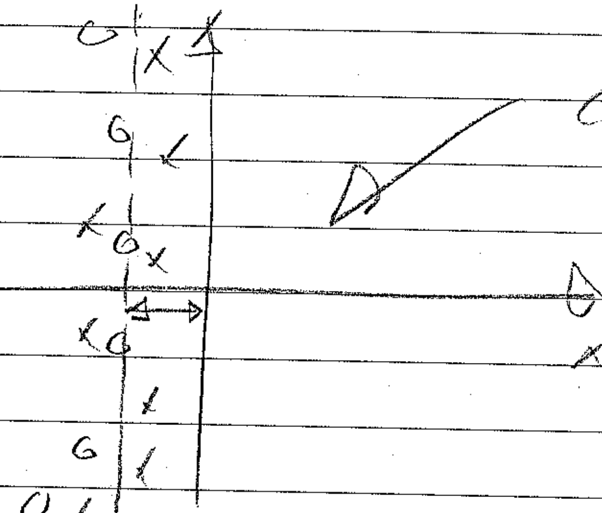
let $g = 0$ and $a_0 + 2\alpha = 0 \Rightarrow$

(*) $\|u\|_t^2 = -a(1)u(1)^2 + \int_0^1 a u_x^2 dx$

(**) $(\|u\|_h^2)_t = -a_N(t)u_N^2(t) + u^T P A u$

(**) Not necessarily strictly stable

6) Since at most right eigenvalue might not be captured except for



Common result

(o) = cont

(x) = discrete

Linear function $a = a_0 + a_1 x \Rightarrow a_x = a_1$

$\Rightarrow (x) \quad \|u\|'_t \le a_1 \|u\| \Rightarrow \|u\| = \|f\| e^{a_1 t}$

$\Rightarrow (xx) \quad (\|u_h\|)'_t \le a \|u_h\| \Rightarrow \|u_h\| \le \|f\|_h e^{a_1 t}$

For higher order polynomials not always true.

See J. Nordström, JSC 2006

Conservative...

Journal of Scientific Computing, Vol. 29, No. 3, December 2006 (© 2005)
DOI: 10.1007/s10915-005-9013-4

Conservative Finite Difference Formulations, Variable Coefficients, Energy Estimates and Artificial Dissipation

Jan Nordström[1]

Received October 22, 2004; accepted (in revised form) September 11, 2005; Published online December 7, 2005

Artificial dissipation terms for finite difference approximations of linear hyperbolic problems with variable coefficients are determined such that an energy estimate and strict stability is obtained. Both conservative and non-conservative approximations are considered. The dissipation terms are computed such that there is no loss of accuracy.

KEY WORDS: Artificial dissipation; finite differences; stability; variable coefficients; energy estimate.

1. INTRODUCTION

Most difference methods for solving nonlinear hyperbolic problems are on conservative form. Conservation is required for a correct shock speed in a nonlinear problem, see [7]. For variable coefficient problems both conservative and non-conservative formulations are used. Examples of important variable coefficient problems in applications include aeroacoustic (the linearized Euler equations), electro-magnetics (variable permittivity and permeability in the Maxwell's equations) and problems where curvilinear meshes with varying metric coefficients are used.

Normally, the artificial dissipation is constructed to absorb the energy of unresolved modes in the problem. It can also be added to enable the calculation of problems involving shocks (see [9] for a discussion on artificial dissipation operators). In this paper we aim for a particular kind of

[1] The Swedish Defence Research Agency, Division of Systems Technology, Computational Physics Department; and the Department of Scientific Computing, Information Technology, Uppsala University, Uppsala, Sweden. E-mail: Jan.Nordstrom@foi.se.

0885-7474/06/1200-0375/0