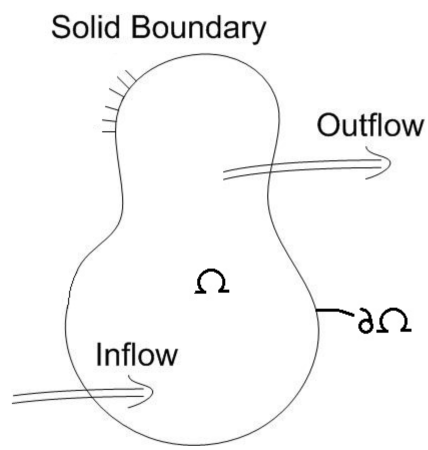
Lecture 1

1 Well-posed problems

General IBVP:



$$\begin{array}{lcllll} u_t & = & Pu + F, & x \in \Omega, & t \geq 0 \\ Bu & = & g, & x \in \Omega, & t \geq 0 \\ u & = & f(x), & x \in \Omega, & t = 0 \end{array} \qquad (1)$$

In (1), P=Diff. operator, B= Boundary operator, F= Forcing function, g= Boundary data, f= Initial data. F, g, f= Data.

The problem (1) is well-posed if u exists and satisfies

$$\|u\|_I^2 \leq K(\|f\|_{II}^2 + \|F\|_{III}^2 + \|g\|_{IV}^2). \tag{2}$$

In (2), $K = K(t \leq T)$ is independent of the data F, f, g; K small is good.

Why is this interesting?
Consider the perturbed problem

$$\begin{array}{lcll} u_t & = & Pu + F + \delta F, & x \in \Omega, \quad t \geq 0 \\ Bu & = & g + \delta g, & x \in \Omega, \quad t \geq 0 \\ u & = & f + \delta f, & x \in \Omega, \quad t = 0 \end{array} \tag{3}$$

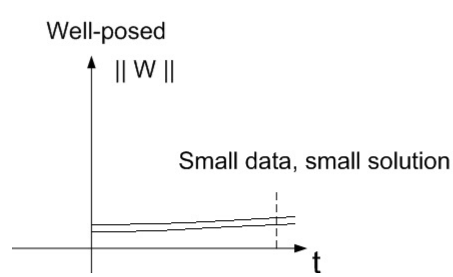
(3)-(1) $\Rightarrow w = (v - u)$, P=lin. op.

$$\begin{array}{lcll} w_t & = & Pw + \delta F, & x \in \Omega, \quad t \geq 0 \\ Bw & = & \delta g, & x \in \Omega, \quad t \geq 0 \\ w & = & \delta f, & x \in \Omega, \quad t = 0 \end{array} \tag{4}$$

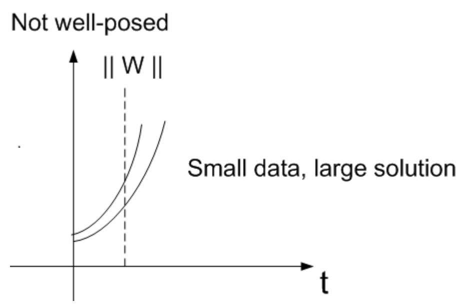
Apply (2) to (4) $\Rightarrow$

$$\|w\|_I^2 \leq K(\|\delta f\|_{II}^2 + \|\delta F\|_{III}^2 + \|\delta g\|_{IV}^2) \tag{*}$$

Now clearly, w=v-u is small if K moderate and perturbations small.



A good numerical approximation possible. Choice of numerical method next step.



A good numerical approximation <u>not</u> possible. <u>No</u> next step! (change problem!).

<u>Note</u>: Uniqueness follows from (*).

<u>**Ex**</u>:

$$\begin{array}{rcll} u_t & = & -u_x + F(x,t), & x \leq 0, \quad t \leq 0 \\ u & = & g(t), & x = 0, \quad t \geq 0 \\ u & = & f(x), & x \leq 0, \quad t = 0 \end{array} \tag{4}$$

$$P = -\delta/\delta x, \;\; \Omega : x \geq 0, \;\; \delta\Omega : x = 0.$$

$$\underbrace{\int_0^\infty u u_t dx}_{\frac{1}{2}\frac{d}{dt}(\|u\|^2)} = \underbrace{-\int_0^\infty u u_x dx}_{-\frac{u^2}{2}|_0^\infty} + \underbrace{\int_0^\infty uT dx}_{\leq \frac{1}{2}(\eta\|u\|^2 + \frac{1}{\eta}\|F\|^2)} \quad \Rightarrow (u = 0 \;\; at \;\; x \;\; \to \;\; \infty)$$

$$\frac{d}{dt}\|u\|^2 \leq \eta\|u\|^2 + g^2 + \eta\|F\|^2 \Rightarrow$$

$$\|u\|^2 \leq e^{\eta T}[\|f\|^2 + \int_0^T e^{-\eta\xi}(g^2 + \frac{1}{\eta}\|F\|^2)d\xi] \tag{5}$$

Compare with (2).

$$K = e^{\eta T},\ \|\|_I^2 = \|\|_{II}^2,\ \|F\|_{III}^2 = \int_0^T e^{-\eta\xi}\frac{1}{\eta}\|F\|_I^2 d\xi,\ \|g\|_{IV}^2 = \int_0^T e^{-\eta\xi} g^2 d\xi$$

Well-posed if u exists (which it does in this case).

An <u>Ill-posed</u> problem (heat equation backwards)

$$\begin{array}{lcll} u_t & = & -u_{xx}, & 0 \le x \le 2\pi, \quad t \ge 0 \\ u(x,0) & = & f(x), & 0 \le x \le 2\pi, \quad t = 0 \end{array} \tag{6}$$

$u = 2\pi$ periodic sol. in space

$$\Rightarrow\ u = \frac{1}{\sqrt{2\pi}} \sum_{w=-\infty}^{\infty} \hat{u}(w,t) e^{iwx}\ \Rightarrow$$

$$\hat{u}_t = w^2 \hat{u} \Rightarrow \hat{u} = e^{w^2 t}\hat{u}(0) = e^{w^2 t}\hat{f}(w) \tag{7}$$

Lemma (Parseval's relation)

Let $u = \frac{1}{\sqrt{2\pi}} \sum_{-\infty}^{\infty} \hat{u}(w,t) e^{iwx}$, then $\|u\|^2 = \int_0^{2\pi} |u|^2 dx = \sum_{-\infty}^{\infty} |\hat{u}(w)|^2$.

Parseval's relation $\Rightarrow$ (7) goes to $\|u\|^2 = \sum_{-\infty}^{\infty} e^{2w^2 t} |\hat{f}|^2 \le K\|f\|^2$

∵ Impossible to bound (by $K < \infty$). Small errors amplified.
∵ <u>Well-posedness very important</u>.

Another Ill-posed problem (even more relevant)

$$\begin{array}{rcl} u_t + u_x & = & 0, \quad 0 \leq x \leq 1, \quad Bu = g. \\ u(x,0) & = & 0, \end{array} \tag{5}$$

Laplace $\Rightarrow s\hat{u} + \bar{u_x} = 0 \Rightarrow \hat{u} = C_1 e^{-sx}$

B=1 (Standard case).
B.C. $\Rightarrow C_1 = \hat{g} \Rightarrow \hat{u} = \hat{g}e^{-sx} \Rightarrow$ well-posed

$B = \alpha + \beta \partial / \partial x$ (Robin cond.)
B.C. $\Rightarrow C_1(\alpha - \beta s) = \hat{g} \Rightarrow \hat{u} = \frac{\hat{g}}{\alpha - \beta s} e^{-sx}$
$s = \frac{\alpha}{\beta}$ singularity $\frac{\alpha}{\beta} \gg 1 \Rightarrow$
Large exponential growth when transforming back. $\Rightarrow$ Ill-posed

2 Well-posedness of non-linear problems

(See Kreiss and Lorenz, IBVPs and the N-S equations, Academic Press 1989)

Linearizion principle: A non-linear problem is well-posed at u if the linear problem obtained by linearizing at all functions near u is well-posed.

Localization principle: If all frozen coefficient problems are well-posed, then the linear problem is also well-posed.

Ex:

$$\begin{array}{rcll} u_t + uu_x & = & 0, & \text{non-linear} \\ u_t + \bar{u}(x,t)u_x & = & 0, & \text{linear} \\ u_t + \bar{u}u_x & = & 0, & \text{frozen coeff.} \end{array} \tag{6}$$

Here there is more research to do!

3 Summary of well-posedness

A problem is well-posed if:

i) A solution exists.

ii) The solution is unique.

iii) The solution can be bounded by the data of the problem.

A nonlinear problem is related to well-posedness through the linearizion and localization principles. (More work to do)

If a problem is not well-posed, do not discretize. Modify first, in practice change B.C.!