# Lecture 2

## 1 Periodic problems and Fourier analysis

How should we define well-posedness?
Consider the IVP (or Cauchy problem) in 1D ($2\pi$-periodic)

$$\begin{array}{lcl} u_t & = & P(\partial/\partial x)u + F, \\ u(x,0) & = & f(x), \end{array} \tag{2.6}$$

P($\alpha u + \beta v$)=$\alpha Pu + \beta Pv$, Linear operator.

<u>**Ex. 1:**</u>

$$P = A\partial/\partial x;\ A = \begin{pmatrix} 0 & 4 \\ 1 & 0 \end{pmatrix}$$

$$T^{-1}AT = \Lambda = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}, T = \begin{pmatrix} 1 & 1 \\ \frac{1}{2} & \frac{-1}{2} \end{pmatrix}, T^{-1} = \begin{pmatrix} \frac{1}{2} & 1 \\ \frac{1}{2} & -1 \end{pmatrix}$$

Rewrite $\Rightarrow v = T^{-1}u, g = T^{-1}f \Rightarrow$

$$\begin{array}{lcl} v_t & = & \Lambda v_x \\ v(x,0) & = & g(x) \end{array}$$

Solution:

$$\begin{aligned} v^{(1)} &= g^1(x+2t) \\ v^{(2)} &= g^2(x-2t) \quad \underline{\text{Exact solution.}} \end{aligned}$$

$$\begin{aligned} ||v|| &\leq ||T^{-1}||\ ||f|| \\ ||u|| &\leq ||T||\ ||v|| \quad \leq \quad \underbrace{||T||\ ||T^{-1}||}_{\kappa=\text{condition number}} \quad ||f|| \end{aligned}$$

**Ex. 2:**

$$\left.\begin{aligned} u_t &= \alpha u \\ u(x,0) &= f(x) \end{aligned}\right\} \Rightarrow \underline{u = e^{\alpha t} f} \text{ exact sol., } \underline{\|u\| \leq e^{\alpha t}\|f\|}$$

**Definition of well-posedness**

The problem (2.6) is well-posed if for F=0, there is a unique solution satisfying

$$\|u\| \leq K e^{\alpha t}\|f\| \tag{2.7}$$

K and $\alpha$ are constants, No zero order terms, $\Rightarrow \alpha = 0$.

If a forcing funtion is present?

$$\|u\|^2 \leq K e^{2\alpha t}(\|f\|^2 + \textstyle\int_0^t \|F\|^2 dt)$$

Proof: Duhamel's principle, see GKO.

$\because$ Very important: Forcing functions do not influence well-posedness.

Existence is generally very difficult.

Uniqueness follows directly from the energy estimate.

Consider (2.6) with $P = \sum_{r=0}^{q} A_r \frac{\partial^r}{\partial x^r}$, F=0, $A_r$ = constant matrics, $u = \frac{1}{\sqrt{2\pi}} \sum_{-\infty}^{\infty} \hat{u}_w e^{iwx}, u = 2\pi$ periodic.

$$\Rightarrow \hat{u} = e^{\hat{P}(iw)t} \hat{f}(w);\ \hat{P}(iw) = \sum_{r=0}^{q} A_r (iw)^r$$

$\hat{P}(iw)$ = symbol or Fourier transform of P.
$e^{\hat{P}(iw)t}$ = matrix exponential.

**Ex.:**

$$u_t + Au_x = Bu_{xx};\ A, B = m \times m,\ P(iw) = -(iwA + w^2 B)$$

**Theory** (2.6) is well-posed if and only if there are constants K and $\alpha$ such that

$$|e^{\hat{P}(iw)t}| \leq Ke^{\alpha t} \quad \text{(2-9)}$$

Proof: $\|u\|^2 = \sum_{-\infty}^{\infty} |\hat{u}^2| = \sum_{-\infty}^{\infty} |e^{\hat{P}(iw)t} \hat{f}|^2 \leq |e^{\hat{P}(iw)t}|^2_{max} \sum_{-\infty}^{\infty} |\hat{f}|^2 = K^2 e^{2\alpha t} \|f\|^2 \blacksquare$

Eigenvalus are easier to work with.

**Def.**: The Petrovski condition is satisfied if the eigenvalues $\lambda(w)$ of $\hat{P}(iw)$ satisfy

$$Re(\lambda(w)) \leq \alpha \qquad (2.10)$$

$\alpha$ = constant indep. of w, $\alpha$ =0 if no zero order terms.

**Theorem** The Petrovski condition is a necessary condition and sufficiant if there is a constant K and matrix T($w$) such that $T^{-1}(w)\hat{P}(iw)T(w)$ is diagonal and $|T^{-1}(w)||T(w)| \leq K$ for all $w$.

Proof

$$\hat{u} = e^{\hat{P}t}\hat{f} = T[e^{-\Lambda t}]T^{-1}\hat{f};$$

$$\|u\|^2 \leq \sum_{\infty}^{\infty} |T|^2|e^{\Lambda t}|^2|T^{-1}|^2|\hat{f}|^2 \leq |T^{-1}|^2|T|^2 \sum_{-\infty}^{\infty} |e^{\Lambda t}|^2|\hat{f}| \leq K^2|e^{\Lambda t}|^2\|f\|^2 \leq K^2 e^{2\alpha t}\|f\|^2 \quad \blacksquare$$

$$(|e^{\Lambda t}|^2 = \text{diag}(e^{\lambda_1 t},\ e^{\lambda_2 t}, ...,\ e^{\lambda_m t}) \leq e^{2\alpha t})$$

Multiple dimensions, exactly similar.

define $\underset{\sim}{x} = (x_0^1, x_1^2, ...., x^\alpha)^T$
$\underset{\sim}{w} = (w^1, w^2, ..., w^\alpha)^T$
$\hat{P}(i\underset{\sim}{w})$ is obtained by $\partial/\partial x^r \rightarrow iw^r$

**Ex**:

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \partial/\partial x + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \partial/\partial y \rightarrow \hat{p}(i\underset{\sim}{w}) = i \begin{pmatrix} w^1 & w^2 \\ w^2 & w^1 \end{pmatrix}$$