Lecture 3

1 Difference approximations

(Very diff. from book)

Consider the approx. of the heat eq.

$$\begin{aligned} \tfrac{d}{dt}u_j &= Qu_j = (u_{j+1} - 2u_j + u_{i+1})/h \\ u_j(0) &= f_j, \qquad j = 0, 1, .., N \quad t \geq 0 \end{aligned} \tag{2.12}$$

Expand in Fourier-series $\Rightarrow$

$$\left.\begin{aligned} \tfrac{d}{dt}(\hat{u}_w) &= \hat{Q}\hat{u}_w \\ \hat{u}_w &= \hat{f}_w \end{aligned}\right. \Rightarrow \hat{u}_w(t) = e^{\hat{Q}t}\hat{f}_w$$

$\because$ Exactly as in cont. case.

We have $Q = D_+D_- \Rightarrow Qe^{iwx_j} = e^{iwx_j}\hat{Q}$

$$\hat{Q} = -\tfrac{4}{h^2}sin^2(\zeta/2);\ \zeta = wh$$

The Von Neumann (Petrovski) condition for stability is $\because$

$$\text{Real}(\hat{Q}) \leq \alpha \text{ indep. of } w.$$

$\alpha = 0$ if no zero order terms.

Note: The Von Neumann condition on the time-step Δt comes in from the specific time-advancement scheme. The eigenvalue condition is more fundamental.

Ex: Euler forward

$$\hat{u}^{n+1} = \underbrace{(1 + \Delta t \hat{Q})}_{\hat{Q}} \hat{u}^n$$

$$|\hat{Q}| \leq 1 \Rightarrow \text{Condition on time-step.}$$

$$|\Delta t \hat{Q} + 1| \leq 0 \Rightarrow \lambda = \tfrac{\Delta t}{h^2} \leq \tfrac{1}{2}$$

$\because$ Semi-discrete analysis very similar to the continuous analysis for well-posedness.

$\because$ Stability in semi-discrete form $\approx$ well-posedness for the PDE.

2 General semi-discrete formulations

$$\begin{aligned} \tfrac{d}{dt} u_j &= Q u_j + F_j \\ u_j(0) &= f_j \end{aligned} \tag{2.25}$$

u_j, F_j and f_j vectors, Q= matrix

Def. The Von Neumaun (Petrovski) condition is satisfied if the eigenvalues of the symbol $\hat{Q}(\zeta)$ satisfy

$$Re(z(\zeta, h)) \leq \alpha, \;|\zeta| \leq \pi \tag{2.26}$$

No terms of order one in Q (zero order terms) then $\alpha = 0$.

<u>Theorem:</u> The problem (2.25) is stable in the semi-discrete sense if (2.26) is valid and $\hat{Q}$ can be diagonalised using a similarity transform with bounded condition number.

<u>Proof:</u>

$$\begin{array}{rcl} (\hat{u}_w)_t &=& \hat{Q}\hat{u}_w + \hat{F} \\ \hat{u}_w(0) &=& \hat{f}_w \end{array} \Rightarrow$$

$$\frac{d}{dt}(e^{-\hat{Q}t}\hat{u}_w) = e^{-\hat{Q}t}\hat{F}$$

$$\hat{u}_w(t) = e^{\hat{Q}t}[\hat{f} + \underbrace{\int_0^t e^{-\hat{Q}\zeta}\hat{F}d\zeta]}_{\text{Ignore now}}$$

$$\hat{u}_w(t) = e^{\hat{Q}t}\hat{f} = T^{-1}e^{\Lambda t}T\hat{f} \Rightarrow$$

$$|\hat{u}|^2 \leq \underbrace{|T^{-1}|^2|T|^2}_{=K^2} \underbrace{e^{2\alpha t}}_{VonNeumann} |\hat{f}|^2 \leq K^2 e^{2\alpha t}|\hat{f}|^2$$

Parsevals relation lead to stability. ■

Forcing function no problem since

$$|\int_0^t e^{-\hat{Q}(t-\zeta)}\hat{F}d\zeta| \leq K|\hat{F}|_{max} \underbrace{|\int_0^t e^{\Lambda(t-\zeta)}d\zeta|}_{\text{Von Neumann}}$$

$$\leq K|\hat{F}|_{max}|\int_0^t e^{\alpha(t-\zeta)}d\zeta| \leq$$

$$\leq K|\hat{F}|_{max}e^{\alpha t}[\tfrac{e^{-\alpha\zeta}}{-\alpha}]_0^t = K|\hat{F}|_{max}(\tfrac{e^{\alpha t}-1}{\alpha})$$

$\because |u|_w^2 \leq K^2 e^{2\alpha t}|\hat{f}|^2 + K^2|\hat{F}|_{max}^2(\frac{e^{\alpha t}-1}{\alpha})$

$\because$ Again just as in cont. case.

3 Summary of IVP's

The continous / semi-discrete problem is well-posed / stable if

i) The Petrovski/Von Neumann condition is satisfied. (Eigenvalue condition).

ii) If the "related" matrix $\hat{Q} = T\Lambda T^{-1}$ can be diagonalized with a similarity transform with bounded condition number k, i.e. $|T||T^{-1}| \leq K$.

Stability in a semi-discrete form
$\approx$
Well-posedness for the PDE.