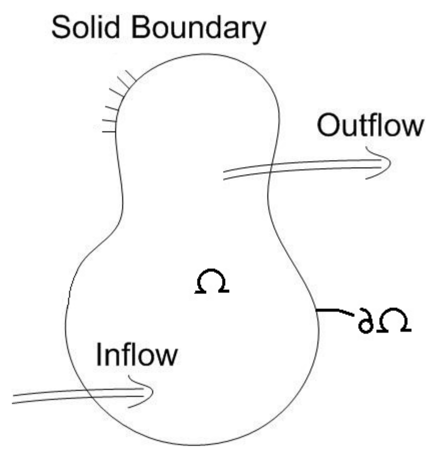
Lecture 4

1 Initial Boundary Value Problem "The Big Picture"



B.C. ? Where, how many, on what form?

$$IBVP \begin{cases} u_t + P(u, (\partial/\partial x))u = F(x,t), & x \in \Omega \\ Lu = g(x,t), & x \in \delta\Omega \\ u(x,0) = f(x), & x \in \Omega \end{cases}$$

IVP = No Boundaries, Periodic. No second row above.

IBVP "Roughly Speaking"

$$\begin{array}{ll} P + (L) \longrightarrow \tilde{P} & \\ & \Rightarrow \\ F + (g) \longrightarrow \tilde{F} & \end{array} \qquad \left. \begin{array}{l} U_t + \tilde{P}u = \tilde{F} \\ \\ u(x,0) = f \end{array} \right. \quad (IVP)$$

$\tilde{P}$, $\tilde{F}$ generalized operator, data.
Eigenvalue analysis (shown later) $\Rightarrow$ $\tilde{P}u = (\Lambda^R + i\Lambda^I)u$

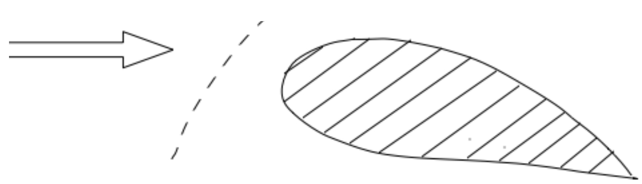
Hyperbolic	$\Lambda^R \approx 0$	(transport, Euler, Maxwell, b.c.=?)
Parabolic	$\Lambda^R > 0$	(damping, heat, diffusion, b.c. everywhere)
Incompletely Parabolic	$\Lambda^R \geq 0$	(N-S, mixed systems, b.c.=?)
Well-posed	$\lvert\Lambda^R\rvert \leq const.$	b.c. !!

i) Must choose L such that $P + L = \tilde{P}$, OK and do not cause explosion. P often given and OK.

ii) Need to choose L such that we have data Lu-g=0.

iii) i) and ii) often in conflict.

<u>**EX**</u>:
$u = u_\infty$
$u_x = 0$
$\alpha u + \beta u_x = u_\infty$



2 Initial Value Problem + Fourier Expansion. (IVP)

- Continuous + semi discrete $\Rightarrow$

$$\begin{aligned} (\hat{u}_w)_t &= \hat{P}(iw)\hat{u}_w + \hat{F} \qquad (*) \\ \hat{u}_w(0) &= \hat{f} \end{aligned}$$

$$\hat{u}_w(t) = e^{\hat{P}(iw)t}\hat{f} + e^{\hat{P}(iw)t}\int_0^t e^{-\hat{P}(iw)\zeta}\hat{F}d\zeta \qquad (**)$$

Knowledge about $\hat{P}(iw)$+Parsevals inequality$\Rightarrow$ Well-posedness or Stability

Easy to analyze since:

- $\hat{P}(iw)$ "Small" matrix

The Fourier Modes "decouple the problem".

- Eigenvalues, eigenvector possible to compute analytically.

3 Initial Boundary Value Problem (IBVP)

- Continuous: cannot be formulated as an ODE, see $(*)$ and $(**)$ above.

- Semi-discrete: can be put up as a semi-discrete system.

$$\begin{aligned} u_t &= Au + F \\ u(0) &= f \end{aligned} \qquad (***)$$

However, the matrix A (corresponding to $\hat{P}(iw)$) is not small.

- A is not "small".

No "decoupling", all grid points included.

- Eigenvalues, eigenvectors almost impossible to compute analytically. Also b.c. often "ruin" the structure of A.

- Other methods and techniques are necessary.

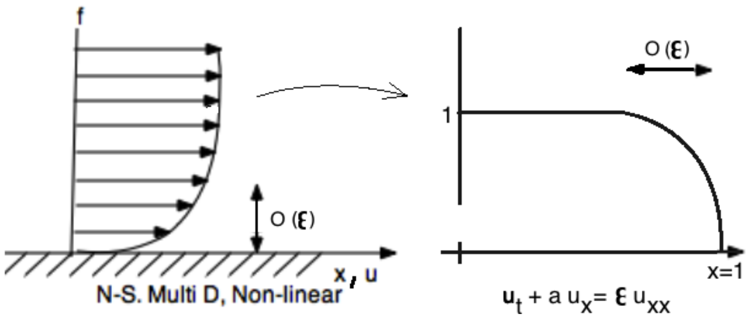
4 Different Levels of Approximations

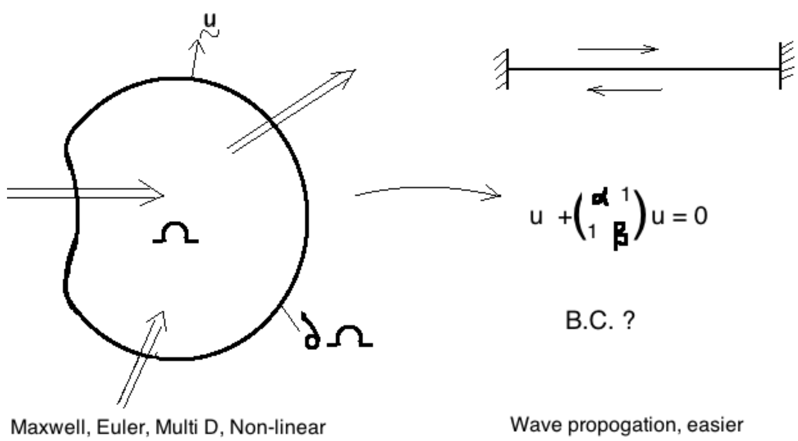
i) Complicated non-linear system of eq's with initial and b.c.'s Ex: N-S with Shocks $\approx$ Impossible to analyze

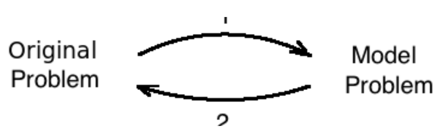
ii) Simplified problem (model problem) with similar character. Possible to analyze

iii) Iterate between i) and ii)

<u>**Model Problems**</u>







5 Initial Boundary Value Problems. Semi-discrete Approximation "The method of lines"

EX:

$$\begin{aligned}
\frac{d}{dt}u_j &= D_0 u_j, \qquad j=1,...,N-1 \\
u_0 &= 2u_1 - u_2 \\
u_N &= g(t) \\
u_j(0) &= f_j, \qquad j=0,...,N
\end{aligned} \tag{2.28}$$

Nh=1, Linear extrapolation, as numerical boundary condition, Not physical!!

Various forms of (2.28), possible variants include (2.29)-(2.31), see the book.

$$\left\{ \begin{array}{l} \frac{du_1}{dt} = \frac{u_2-(2u_1-u_2)}{2\Delta x} = \frac{u_2-u_1}{\Delta x} \\ \\ \frac{du_{N-1}}{dt} = \frac{g-(u_{N-2})}{2h} = -\frac{u_{N-2}}{\Delta x} + \frac{g}{2h} \end{array} \right. \quad \Rightarrow \quad \left. \begin{array}{l} \frac{d}{dt}u = Qu + F \\ \\ u(0) = f \end{array} \right\} \tag{2.19}$$

The general form is:

$$\begin{aligned}
\frac{du_j}{dt} &= Qu_j + F_j, \quad j=1,...,N-1 \\
B_n u &= g \\
u_j(0) &= f_j, \qquad j=0,...,N
\end{aligned} \tag{2.32}$$

$j=1,...,N-1$, inner points. $B_n u = g$. Complete set of boundary conditions (real + numerical).

$B_n u = g$ include as many conditions that are needed for the ODE system to have a unique solution. The number of boundary conditions is equal to the number of linearly independent conditions (No problem with existence!)

The discrete scalar product and norm in analogy with the continuous case is:
$\longrightarrow (u, v)_h = \sum_{j=1}^{N-1} q_j < u_j, \hat{H} v_j > h, \ ||u||^2 = (u, u)_h$

$q_j > 0$ and $\tilde{H}$ pos. def. symmetric matrix. Numbering may vary, may include also boundary points.

<u>**Def.**</u>: Let V_n be space of grid vector functions v satisfying boundary condition $B_n V = 0$. The difference operator Q is semi-bounded if for all v $\in V_n$

$$(v, \ qv)_n \leq \alpha ||v||_n^2$$

<u>**Example**</u>:

$$\begin{array}{lcl} \frac{d}{dt} u_j & = & D u_j, \quad j = 0, 1, ..., N-1 \\ u_N(t) & = & 0 \\ u_j(0) & = & f, \qquad j = 0, ..., N \end{array}$$

$$D u_j = \begin{cases} D_+ u_j & j = 0 \\ D_0 u_j & j = 1, 2, ..., N-1 \end{cases}$$

Redefine scalar product as

$$(u, v)_h = \delta h u_0 v_0 + \sum_{j=1}^{N-1} u_j v_j h = u^T P v.$$

Then

$$(u, Qu) = (u, Du) =$$

$$\delta h u_0 (Du)_0 + \sum_{j=1}^{N-1} u_j (\frac{u_{j+1} - u_{j-1}}{2h}) h = \delta(u_0 u_1 - u_0^2) + u_1 \frac{u2 - u_0}{2} + u_2 \frac{u_3 - u_1}{2} + ...$$

$$u_{N-2}(u_{N-1} - uN - 3) + u_{N-1}(u_N - u_{N-2}) = -\delta u_0^2 + u_0 u_1 (\delta - \frac{1}{2})$$

$$= -\frac{1}{2} u_0 \text{ if } \delta = \frac{1}{2}$$

$\because$ Q semi-bounded (SBP trick!!)

Continuous Problem: Semi-boundedness is shown using integration-by-parts

Semi-discrete Problem: Semi-boundedness is shown by summation-by-parts

Definition:
The problem (2.32) is stable if for F=g=0, $||u||_h \leq k e^{\alpha t} ||f||_h$ holds. k and α are constants independent of f and h.

Note that the constants have to be independent of h. The estimate must be independent of grid.

Theorem: If Q is semi-bounded, then (2.32) is stable.

Note! No problem with existence and number of boundary conditions and maximal semi-boundedness etc.
Note also!! A non-zero forcing function F is of no problem if $||F||$ bounded, see ex (2.29) in book.

Definition The problem 2.32 is strongly stable if it satisfies

$$||u||^2 \leq Ke^{\alpha t}(||f||^2 + \int_0^t (||F||^2 + ||g||^2)dt)))$$
$$(\int_0^T e^{-\alpha\tau}(\frac{1}{\alpha}||F||^2 + g^2)d\tau)$$

K and α are constants independent of F, f, g, h.

6 Time-stability

The formal stability definitions allow for exponential growth in time for fixed h, k. (only accurate in the limit)

Boundedness in time is very useful for long-time calculations, we will use the concept <u>time stable</u>.

<u>Definition in the book</u>

The problem (2.32) is time stable if for $F = g = 0$, there is a unique solution satisfying

$$||u||_h^2 \leq k||f||_h$$

k independent of f, h, t.

<u>A better definition</u>

Assume that the PDE for $F = g = 0$ has the estimate

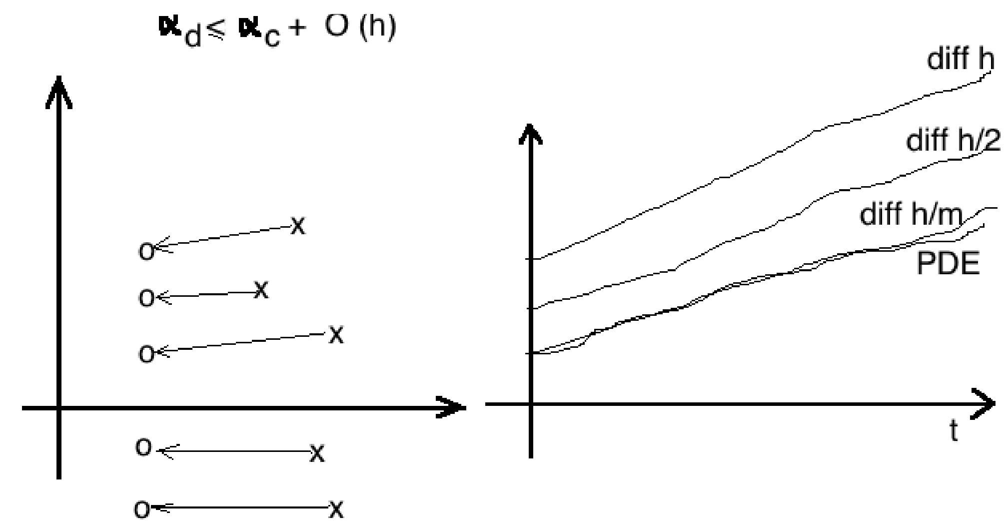
$$||u|| \leq k_c e^{\alpha_c t}||f||$$

The difference approximation (2.32) is time-stable if it has the estimate

$$||u||_h^2 \leq k_d e^{\alpha_d t}||f||_h$$

where $\alpha_d \leq \alpha_c + \mathcal{O}(h)$.

We will come back to time-stability later once we have developed more theory and skill.



Ex: Time stability (periodic case)

Continuous:

The PDE is an skew-symmetric form

$$\begin{aligned} u_t &= p(u) = (au)_x + a(x)u_x,\ a > 0 \\ u(1,t) &= u(0,t) \\ u(x,0) &= f(x) \end{aligned}$$

$$(u, pa) = (u, (au)_x + au_x) = \int_0^1 u(au)_x + auu_x dx = \underbrace{au^2|_0^1}_{=0\ periodic} + \underbrace{\int_0^1 -auu_x + auu_x dx}_{=0} \leq 0$$

$\because$ P Semi bounded $\Rightarrow \frac{d}{dt}||u||^2 = 0$

Semi discrete

$$(u_j)_t = Qu_j = D_0(a_j u_j) + a_j D_0 u_j$$

$x_j = jh,\ j = 0, ..., N,\ Nh = 1,\ u_j = u_j + N?,\ \text{all } j.$

$$(u, v) = \sum_{j=0}^{N-1} u_j v_j h$$

$$(u, D_0 v)_h = \frac{1}{2}\sum_{j=0}^{N-1} u_j(u_{j+1} - u_{j-1}) =$$

$$= \frac{1}{2}\sum_{j=1}^{N} u_{j-1}v_j - \frac{1}{2}\sum_{j=-1}^{N-2} u_{j+1}v_j$$

$$= \frac{1}{2}\sum_{j=0}^{N-1} u_{j-1}v_j - \frac{1}{2}\sum_{j=0}^{N-1} u_{j+1}v_j = -\frac{1}{2}\sum_{j=0}^{N-1}(u_{j+1} - u_{j-1})v_j = -(D_0 u, v)$$

$\because D_0$ is skew-symmeric

$$(u, Qu) = \underbrace{(u, D_0(au))}_{=-(D_0 u, au)} + (u, aD_0 u) = 0$$

$$\frac{d}{dt}||u||_h^2 = 0$$

$$\frac{du_j}{dt} = Qu_j \text{ time-stable}$$

$$(a, D_0(Au)) + (u, AD_0 u) = -(D_0 u, Au) + (u, AD_0 u) = -(D_0 u)^T + Au + u^T AD_0 u = 0$$