Lecture 5

1 IBVP's and the energy-method

1.1 The PDE

Ex:

$$\begin{aligned} u_t &= (au_x)_x; \quad 0 \le x \le 1; \quad t \ge 0 \\ u(0,t) &= 0 \\ u_x(1,t) &= 0 \\ u(x,0) &= f(x) \end{aligned}$$

$a = a(x,t) \ge \delta > 0.$

Define scalar product and norm.

$$(u,v) = \int_0^1 uv dx, \ \ ||u||^2 = (u,v)$$

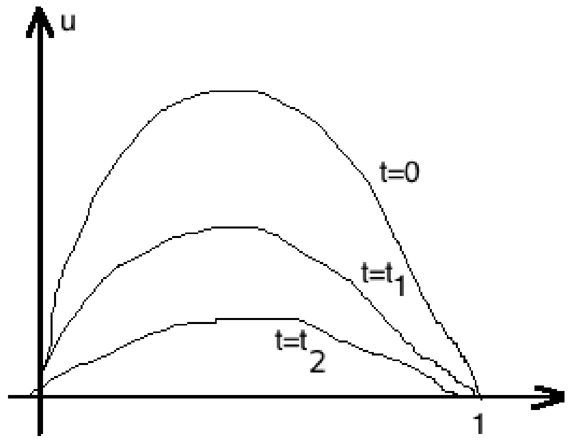
Energy-method, multiply and integrate $\Rightarrow$

$$\int_0^1 uu_t dx = \int_0^1 u(au_x)_x dx = uau_x|_0^1 - \int_0^1 au_x^2 dx$$

$$\frac{d}{dt}||u||^2 = \underbrace{2auu_x|_1}_{=0} - \underbrace{2auu_x|_0}_{=0} - 2(u_x, au_x) \le -2\delta ||u_x||^2$$

Integration yields

$||u||^2 + 2\delta \int_0^t ||u_x||^2 d\tau \le ||f||^2$



u decays if u_x nonzero, typical for parabolic equations, like the heat and diffusion equations.

The operator $p = \frac{\partial}{\partial x} a \frac{\partial}{\partial x}$ is semi-bounded, i.e:

$$(u, pu) \leq 0 \qquad (-\delta ||u_x||^2)$$

We will come back to this.

Consider the general Initial Boundary Value problem (2.1)

$$\begin{array}{rcll} u_t &=& pu + F; & 0 \leq x \leq 1 \\ Bu &=& g; & x = 0, 1 \\ u &=& f; & 0 \leq x \leq 1 \end{array} \tag{2.1}$$

Define new scalar products and norms as

$$(u, v) = \int_0^1 quHv dx, \;\; ||u||^2 = (u, u)$$

$q(x) > 0$, $H(x)$ pos. def. Hermitian matrix. (Normally $q = 1$, $H = I$)

Definition

Let V be the space of differentiable functions satisfying the homogeneous b.c $BU = 0$. The differentiable operator P is semi-bounded if for all $u \in V$, $(u, Pu) \leq \alpha ||u||^2 - \delta ||u_x||^2,\ \alpha = \text{constant}$.

If a solution exists, semi-boundedness guarantee well posedness since

$$\frac{d}{dt}||u||^2 = 2(u, u_t) = \underbrace{2(u, Pu)}_{\leq 2\alpha ||u||^2} \Rightarrow ||u||^2 \leq e^{2\alpha t}||f||^2$$

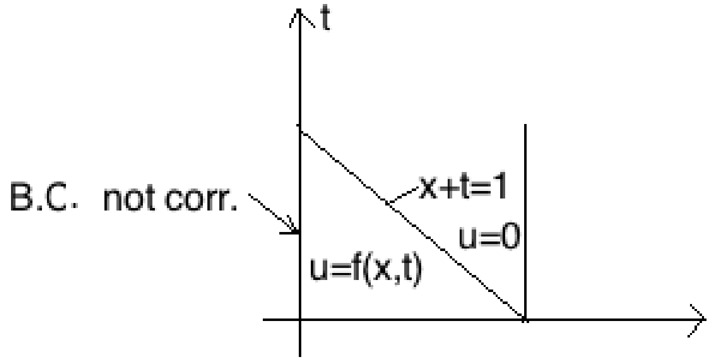
Remark: The boundary term must have the correct sign, no way to estimate $|u|^2_\infty$ in term of $||u||$. To do that we need

$$|u|^2_\infty \leq \theta ||u||^2 + \theta^{-1}||u_x||^2$$

Existence:

$$\begin{aligned} u_t &= u_x; \quad 0 \leq x \leq 1 \\ u(0,t) &= 0 \\ u(1,t) &= 0 \\ u(x,0) &= f(x) \end{aligned}$$

$(u, Pu) = 0 \because$ semi-bounded operator.
However, the boundary condition at 0 is not correct.



$$u = \begin{cases} f(x+t) & x+t \leq 1 \\ 0 & x+t > 1 \end{cases}$$

$u(0,t) = f(t)$ for $t \leq 1$, if f not identically zero, $u(0,t) = 0$ contradicts that.
$\because$ No existence, must restrict semi-boundedness.

Definition:
P is <u>maximally semi-bounded</u> if it is semi-bounded in V but not semi-bounded in any space with fewer (minimally) boundary conditions.

In our example V is to too "small" for allowing the existence of solution, must be made "bigger" by dropping the boundary condition at x=0.

Choose v={u(x,t), u(1,t)=0} $\Rightarrow (u, Pu) = -\frac{u(0,t)^2}{2} \leq 0 \longleftarrow$ just right.

Choose u={u(x,t)} $\Rightarrow$ V too "large"

$$(u, Pu) = \frac{u(1,t)^2}{2} - \frac{u(0,t)^2}{2} \quad \text{not bounded.}$$

Maximal semi-boundedness implies well-posedness.

Definition:
The IBVP (2.1) is well-posed if for $F = q = 0$, i.e there is a unique solution satisfying

$$||u|| \leq ke^{\alpha t}||f||$$

k and α are constants independent of f.

Ex:

$$\begin{aligned} u_t &= u_x \\ u(1,1) &= g(t) \\ u(x,0) &= f(x) \end{aligned}$$

Energy method yields

$$||u||_t^2 = g^2 - u(0)^2 \Rightarrow$$

$$||u||^2 + \int_0^t u(0,t)^2 dt = ||f||^2 + \int_0^t g^2(t)dt$$

Definition:
The IBVP (2.1) is strongly well-posed if there is a unique solution satisfying:

$$||u||^2 \leq ke^{\alpha t}[||f||^2 + \int_0^t [||F||^2 + g^2]d\tau]$$

k and α are constants independent of $F,\ f,\ g$.

EX: (Number of B.C's) $P = A\frac{\partial}{\partial x}$

$$u_t = Au_x;\ 0 \leq x \leq 1$$

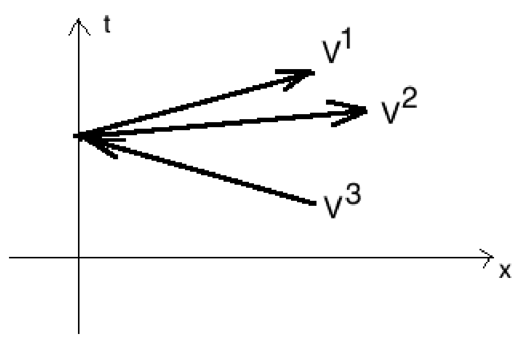
$$T^{-1}AT = \Lambda = \begin{bmatrix} \lambda_1 & & & & & \\ & \ddots & & & & \\ & & \lambda_r & & & \\ & & & \lambda_{r+1} & & \\ & & & & \ddots & \\ & & & & & \lambda_m \end{bmatrix}$$

$$V = T^{-1}u = \begin{pmatrix} V^I \\ V^{II} \end{pmatrix} \Rightarrow$$

$$V_t^j = \lambda_j V_x$$

Ex

m=3; r=2, $\lambda_1, \lambda_2 < 0,\ \lambda_3 > 0$



$$X = 0: \begin{array}{l} V^1 = S_1 V^{(3)} + g_1 \\ V^2 = S_2 V^{(3)} + g_2 \end{array} \Rightarrow (V^I = S^I V^{II} + g^I)$$

$$X = 1: \quad V^3 = S_3 V^{(1)} + S_4 V^{(2)} + g_3 \quad \Rightarrow (V^{II} = S^{II} V^I + g^{II})$$

$$\Rightarrow$$

$$(1)\begin{cases} V_t &= \Lambda V_x; & 0 \le x \le 1 \\ V^I &= S^I V^{II} + g^I, & x = 0 \\ V^{II} &= S^{II} V^I + g^{II}, & x = 1 \\ \underset{\sim}{V} &= \tilde{f} \end{cases}$$

$$S^I = \left.\begin{bmatrix} \cdot \\ \cdot \end{bmatrix}\right\} r$$

$$S^{II} = [\ .\quad .\ ]\ \} m - r$$

$$(2)\begin{cases} u_t &= A u_x \\ B^I u &= \tilde{q}^I \\ B^{II} u &= \tilde{q}^{II} \\ \underset{\sim}{u} &= \tilde{f} \end{cases}$$

$$B^I = r \underbrace{\begin{bmatrix} \qquad\qquad \end{bmatrix}}_{\text{m}}$$

$$B^{II} = (m-r) \underbrace{[\qquad\quad]}_{\text{m}}$$

If transform of (2) gives (1) $\Rightarrow$ <u>strongly well-posed</u>, Need special norm.

1.2 Summary

Consider (2.1)

• A maximally semi-bounded operator leads to well-posedness for homogeneous b.c.

• A forcing condition can also be included without problems.

• Strong well-posedness requres further analysis. No general procedure for energy-method, normal mode analysis required.

• First order hyperbolic problems with correct number of b.c's are strongly well-posed.

2 Splitting techniques for the energy-method

Consider an infinite (or periodic) domain (Boundaries can be included and handled).

Linear

$$u_t + au_x = 0;\ \ a = a(x,t) \tag{1}$$

Energy:

$$\int_{-\infty}^{\infty} uu_t dx + \int_{-\infty}^{\infty} auu_x dx = 0 \Rightarrow ||u||_t^2 + au^2|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} a_x u^2 dx = 0$$

$$\because\ ||u||_t^2 = \int_{-\infty}^{\infty} a_x u^2 dx \leq |a|_x ||u||^2$$

Semi-bounded, Well-posed

Semi-discrete Q=differential operator

$$u_t + AQu = 0,\ \ Q + Q^T = 0 \quad \text{skew-symmetric.}$$

Energy: Let $I = u^T u_t \Delta x + u^T AQu\Delta x = 0 \Rightarrow$

$$\underbrace{2u^T u_t \Delta x}_{||u||_t^2} + \underbrace{u^T(AQ + (AQ)^T)u}_{\neq -2\int_{-\infty}^{\infty} a_x u^2 dx} = 0$$

Split op. using all combinations $\rightrightarrows$ Back to the PDE!

$$au_x = \alpha(au)_x + \beta au_x + \gamma a_x u = au_x(\alpha+\beta) + a_x u(\alpha+\gamma) \Rightarrow$$

$$\begin{matrix} 1 = \alpha + \beta \\ \\ 0 = \alpha + \gamma \end{matrix} \ \Rightarrow\ \begin{matrix} \beta = 1 - \alpha \\ \\ \gamma = -\alpha \end{matrix} \ \Rightarrow$$

$$au_x = \alpha(au)_x + (1-\alpha)au_x - \alpha a_x u$$
$$\int_{-\infty}^{\infty} uu_t dx + \int_{-\infty}^{\infty} u(\alpha(au)_x + (1-\alpha)au_x - \alpha u_x u)dx = 0$$
$$\frac{1}{2}||u||_t^2 + \int_{-\infty}^{\infty} \alpha u(au)_x + (1-\alpha)auu_x = \int_{-\infty}^{\infty} \alpha a_x u^2 dx$$
$$\frac{1}{2}||u||_t^2 + \underbrace{\alpha uau|_{-\infty}^{\infty}}_{=0} + \underbrace{\int_{-\infty}^{\infty} -(1-2\alpha)auu_x dx}_{=0 \text{ if } \alpha=\frac{1}{2}} = \int_{-\infty}^{\infty} \alpha a_x u^2 dx$$
$$\because ||u||_t^2 = \int_{-\infty}^{\infty} a_x u^d x$$

(Which we already know)

Semi-discrete

$$u_t + \frac{1}{2}Q(Au) + \frac{1}{2}AQu - \frac{1}{2}A_x u = 0$$
$$2u^T u_t \Delta x + (u^T QAu + u^T AQu)\Delta x = u^T A_x u \Delta x$$
$$= -u^T Q^T Au + u^T A^T Qu$$
$$= -(Qu)^T (Au) + (Au)^T Qu$$
$$=0$$
$$||u||_t^2 = u^T A_x u \Delta x \sim \int_{-\infty}^{\infty} a_x u^d x \because OK$$

Non-linear

$$u_t + uu_x = 0$$

$$u_t + (\frac{u^2}{2})x = 0$$

$$u_t + \beta u u_x + (1-\beta)(\frac{u^2}{2})_x = 0$$

$$\int_{-\infty}^{\infty} u u_t dx + \int_{-\infty}^{\infty} \underbrace{\beta u^2 u_x}_{\beta u^3|_{-\infty}^{\infty} - \int \beta (u^2)_x u dx} + (1-\beta) u (\frac{u^2}{2})_x dx = 0$$

$$\frac{1}{2}||u||_t^2 + \int_{-\infty}^{\infty} u(u^2)_x \underbrace{(-\beta + \frac{1-\beta}{2})}_{=0 \ if \ \beta = \frac{1}{3}} dx = 0$$

$$\because u_t + \frac{1}{3} u u_x + \frac{1}{3}(u^2)_x = 0 \text{ Appropriate!}$$

<u>**Semi-discrete**</u>:

$$\bar{U} = diag(u_i)$$

$$u_t + \frac{1}{3}\bar{U}Qu + \frac{1}{3}Q(\bar{U}u) = 0$$

$$u^T u_t \Delta x + \frac{1}{3}(u^T \bar{U} Q u + u^T Q \bar{U} u)\Delta x = 0$$

$$\underbrace{\qquad\qquad -u^T Q^T \bar{U} u}_{(\bar{U}u)^T Q u - (Qu)^T \bar{U} u}$$

$$=0$$

$$||u||_t^2 = 0$$

3 Matrix properties

Linear:

$$u_t + \frac{1}{2}\underbrace{(QA + AQ)}_{\bar{Q}(u)} u - \underbrace{\frac{1}{2}A_x u}_{symmetric \ part} = 0$$

Non-linear:

$$u_t + \frac{1}{3}(Qu + uQ)u = 0$$

$$\bar{Q}(u)$$

$$\bar{Q}+\bar{Q}^{=}QA+AQ+(QA)^T+(AQ)^T = QA+AQ+AQ^T+Q^TA = QA+AQ-AQ-QA = 0$$

See JSC vol 29, No3 2006, Conservative finite ... J. Nordström