Lecture 6

1 Stability for method of lines

$$\begin{array}{lcl} u_t + Au & = & 0 \\ u(0) & = & f \end{array} \tag{1}$$

Solution

$$u = e^{-At} f \tag{2}$$

<u>**Eigenvalues**</u>

$$AX = X\Lambda;\ A = X\Lambda X^{-1} \Rightarrow$$

$$u = (I - At + (At)^2/2 + \ldots\ )\ f =$$

$$X\ (I - \Lambda t + (\Lambda t)^2/2 + \ldots\ )\ X^{-1} f =$$

$$X e^{-\Lambda t} X^{-1} f \Rightarrow X^{-1} u = e^{-\Lambda t}\ X^{-1} f$$

$$V(t) = e^{-\Lambda t} V(0)$$

$$e^{-\Lambda t} = \begin{bmatrix} e^{-\lambda t} & & & \\ & e^{-\lambda_2 t} & & \\ & & \ddots & \\ & & & e^{-\lambda_n t} \end{bmatrix}$$

Well-posed/stable if all eigenvalues and eigenvectors OK, $Re(\lambda_i) \geq 0$ or for bounded growth $Re\lambda \geq -\alpha$, $\alpha = $ const.

+ Eigenvalue analysis is very percise.
- Technology very difficult. (GKS, Normal Mode, Laplace)

Energy

$$u^T P u_t + u^T P A u = 0, \; P > 0$$

$$\frac{1}{2}(||u||_P^2)_t + u^T \underbrace{(\frac{PA + (PA)^T)}{2}}_{\tilde{A}} u = 0$$

Well-posed/stable if $\tilde{A}$ positive semi-definite.

+ Technically easy. (Skew-symmetric part is gone)
- Not precise, could be stable anyway.

Example:

Eigenvalues

$$A = \begin{bmatrix} 1 & \Theta \\ 0 & 2 \end{bmatrix}, \ \Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\lambda = 1 \Rightarrow \begin{bmatrix} 0 & \Theta \\ 0 & 1 \end{bmatrix} \quad x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$\lambda = 2 \Rightarrow -x_1 + \Theta x_2 = 0, \quad x_1 = \Theta x_2$

$$\underset{\sim}{x} = \begin{bmatrix} \Theta \\ 1 \end{bmatrix} \frac{1}{\sqrt{1+\Theta^2}}, \ \bar{X} = \begin{bmatrix} 1 & \frac{\Theta}{\sqrt{1+\Theta^2}} \\ 0 & \frac{1}{\sqrt{1+\Theta^2}} \end{bmatrix} |x| \neq 0$$

$\because$ All OK and stable.

Energy

$$u^T u_t + u^T A u = 0 \Rightarrow \tfrac{1}{2}||u||_t^2 + u^T \underbrace{\frac{(A + A^T)}{2}}_{\tilde{A}} u = 0$$

$$\tilde{A} = \begin{bmatrix} 1 & \frac{\Theta}{2} \\ \frac{\Theta}{2} & 2 \end{bmatrix}$$

Eigenvalues of $\tilde{A}$?

$$(1-\lambda)(2-\lambda)-(\tfrac{\Theta}{2})^2 = 0 \Rightarrow \lambda^2-(1+2)\lambda+2-\tfrac{\Theta^2}{2} = 0 \Rightarrow$$

$$\lambda = \tfrac{3}{2} \pm \sqrt{(\tfrac{3}{2})^2-(2-(\tfrac{\Theta}{2})^2)}$$

Stable if $2-(\frac{\Theta}{2})^2 \geq 0 \Rightarrow$

$$(\tfrac{\Theta}{2})^2 \leq 2 \Rightarrow \Theta^2 \leq 8 \quad \therefore \ |\Theta| \leq \sqrt{8}$$

$\therefore$ Energy-estimate if $|\Theta| \leq \sqrt{8}$

$\because$ But stable anyway. (Normal mode, GKS)

$\therefore$ Energy-estimates are conservative.

Can we choose P to fix this?

A simple P is $P = \begin{bmatrix} \alpha & 0 \\ 0 & \gamma \end{bmatrix}$

$$PA = \begin{bmatrix} \alpha & 0 \\ 0 & \gamma \end{bmatrix} \begin{bmatrix} 1 & \Theta \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} \alpha & \alpha\Theta \\ 0 & 2\gamma \end{bmatrix}$$

$$PA + (PA)^T = \begin{bmatrix} \alpha & \frac{\alpha\Theta}{2} \\ \frac{\alpha\Theta}{2} & 2\gamma \end{bmatrix}$$

$$(\alpha-\lambda)(2\gamma-\lambda)-(\frac{\alpha\Theta}{2})^2 = 0$$

$$\lambda^2 - (\alpha + 2\gamma)\lambda + 2\alpha\gamma - (\frac{\alpha\Theta}{2})^2 = 0$$

$$\alpha + 2\gamma > 0, \quad 2\gamma\alpha - \frac{\alpha^2\Theta^2}{2} \geq 0 \Rightarrow$$

$$\alpha^2\Theta^2 \leq 8\alpha\gamma \Rightarrow \Theta^2 \leq 8\frac{\gamma}{\alpha}$$

$\because$ if $\frac{\gamma}{\alpha} \gg 1 \Rightarrow$ non-existant instability effectively removed.

$\because$ Make sure to look for stability in the right norm.