Exercises I

Consider the scalar initial value problem

$$\begin{aligned} u_t &= a u_{xx} + b u_x + c u, \quad t \geq 0\,, \\ u(x,0) &= f(x)\,, \end{aligned} \tag{1}$$

where $f(x)$ is 2π-periodic.

1. Prove that (1) is well posed if and only if there is a constant α such that

$$Re\,(-a\omega^2 + i\omega b + c) \leq \alpha$$

   for all real ω.

2. Assume that $a < 0$ is real. We know from class that the problem is ill posed if $b = c = 0$. Are there any values of b and c that makes the problem well posed?

3. Assume that $Re\, a > 0$. Prove that (1) is well posed for all values of b and c.

4. Assume that $Re\, a = 0$. Prove that (1) is not well posed if $Im\, b \neq 0$. What is the most general well posed form of the PDE if $Re\, a = 0$?

5. Consider the first order system $u_t = Au_x$, where A is a constant matrix. Is it possible that the Petrovskii condition is satisfied for some constant $\alpha > 0$, but not for $\alpha = 0$?

6. Consider the PDE system

$$u_t = Au_x + Bu$$

   where A and B are real constant matrices. What is the condition on A and B for an energy conserving system, i.e., $||u(\cdot,t)|| = ||u(\cdot,0)||$?