Exercises III

1. Find the most general boundary conditions for the hyperbolic system

$$u_t = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} u_x \,, \quad 0 \le x \le 1 \,,\ t \ge 0$$

by diagonalizing the system and then transforming back.

2. Let the scalar product and norm be defined by

$$(u,v)_{r,s} = \sum_{j=r}^{s} u_j v_j h \,, \qquad ||u||^2_{r,s} = (u,u)_{r,s} \,,$$

and use the notation $u_j|_r^s = u_s - u_r$. Prove the equalities

$$\begin{aligned} (u, D_+ v)_{r,s} &= -(D_- u, v)_{r+1,s+1} + u_j v_j|_r^{s+1} \\ &= -(D_+ u, v)_{r,s} - h(D_+ u, D_+ v)_{r,s} + u_j v_j|_r^{s+1} \,, \\ (u, D_0 v)_{r,s} &= -(D_0 u, v)_{r,s} + \frac{1}{2}(u_j v_{j+1} + u_{j+1} v_j)_{r-1}^{s} \,. \end{aligned}$$

3. Find the stability condition in the form $\lambda = k/h \le \lambda_0$ for the approximation

$$\begin{aligned} u_j^{n+1} &= (I + kD_+) u_j^n \,, \quad j = 0, 1, \ldots, N-1 \,, \\ u_N^n &= g^n \,, \\ u_j^0 &= f \end{aligned} \tag{1}$$

by using the energy method ($g^n = 0$). Is λ_0 the same constant as the one obtained from the von Neumann condition?

4. Prove that the solution to (1) satisfies

$$||u^n||_{0,N-1} \le ||f||^2_{0,N-1} + \sum_{\nu=0}^{n-1} |g^n|^2 k \,.$$

5. Use the energy method to prove that

$$\begin{aligned} (I - kQ) u_j^{n+1} &= u_j^n \,, \quad j = 0, 1, \ldots, N-1 \,, \\ u_N^n &= g^n \,, \\ u_j^0 &= f \end{aligned}$$

is unconditionally stable if

$$Qu_j = \begin{cases} D_0 u_j \,, & j = 1, 2, \ldots, N-1 \\ D_+ u_j \,, & j = 0 \end{cases}$$