Exercises V

1. Consider the equation $u_t = a_x$ and the difference scheme

$$\frac{du_j}{dt} = aD_0(I - \frac{h^2}{6}D_+D_-)u_j\,, \quad j = 1, 2, \ldots$$

Use normal mode analysis to show that it is strongly stable with the boundary conditions

$$u_{-1} = g_{-1}$$
$$u_0 = g_0$$

for any value of a.

2. Consider the the equation $u_t + u_x = 0$ with complex solutions and the approximation

$$\frac{du_j}{dt} + D_0 u_j = 0\,, \quad j = 1, 2, \ldots$$

with boundary condition

$$au_0 - u_1 = 0$$

where a is a complex constant. Define the domain

$$\Omega = \{z,\ |z| \leq 1,\ Re\, z \geq 0\}$$

in the complex plane, and apply the normal mode analysis. Prove the following:

- If a belongs to the interior of Ω, then there is an eigenvalue $\tilde{s}$ with $Re\,\tilde{s} > 0$.
- If a does not belong to Ω, then the Kreiss condition is satisfied.
- If $a = i\alpha,\ -1 < \alpha < 1,\ \alpha \neq 0$, then there is an eigenvalue $\tilde{s}$ with $Re\,\tilde{s} = 0$.
- If $|a| = 1,\ Re\, a \geq 0$, then there is a generalized eigenvalue with $Re\,\tilde{s} = 0$.

3. Consider the difference scheme

$$u_j^{n+1} = u_j^n + kaD_- u_j^n\,, \quad j = 1, 2, \ldots$$
$$(u_0^{n+1} + u_1^{n+1})/2 = g^{n+1}$$

where a is a real constant. Derive conditions on a and $\lambda = k/h$ such that the scheme is strongly stable.