A Roadmap to Well Posed and Stable Problems in Computational Physics

By

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$\vec{H}_2(Lu - h_2)$

Well posed problems

$$\begin{aligned} W_t + \mathcal{P}W &= \mathbf{F}, \quad \mathbf{x} \in \Omega, \quad t \geq 0 \\ \mathcal{L}W &= \mathbf{g}, \quad \mathbf{x} \in \partial\Omega, \quad t \geq 0 \\ W &= \mathrm{f}, \quad \mathbf{x} \in \Omega, \quad t = 0 \end{aligned}$$

Definition 1. The initial boundary value problem with F=g=0 is well posed if a unique smooth solution exists that satisfies the estimate

$$\|W(\cdot,t)\|_\Omega^2 \leq K_1^c(t)\|\mathbf{f}\|_\Omega^2$$

Definition 2. The initial boundary value problem is strongly well posed, if it is well-posed and satisfies

$$\|W(\cdot,t)\|_\Omega^2 \leq K_2^c(t)\left(\|\mathbf{f}\|_\Omega^2 + \int_0^t (\|\mathbf{F}(\cdot,\tau)\|_\Omega^2 + \|\mathbf{g}(\tau)\|_{\partial\Omega}^2)d\tau\right)$$

Stable problems

$$\begin{aligned} (W_j)_t + QW_j &= \mathbf{F}_j, \quad \mathbf{x}_j \in \Omega, \quad t \geq 0 \\ \mathcal{M}W_j &= \mathbf{g}_j, \quad \mathbf{x}_j \in \partial\Omega, \quad t \geq 0 \\ W_j &= \mathbf{f}_j, \quad \mathbf{x}_j \in \Omega, \quad t = 0. \end{aligned}$$

Definition 3. The semi-discrete approximation with $\mathbf{F}_j = \mathbf{g}_j = 0$ is stable for every $\mathbf{f}_j$ if the solution satisfies the estimate

$$\|W_j(t)\|^2_{\Omega_h} \leq K_1^d(t)\|\mathbf{f}_j\|^2_{\Omega_h}$$

Definition 4. The semi-discrete approximation is strongly stable, if it is stable and satisfies

$$\|W_j(t)\|^2_{\Omega} \leq K_2^d(t)\left(\|\mathbf{f}_j\|^2_{\Omega_h} + \int_0^t (\|\mathbf{F}_j(\cdot,\tau)\|^2_{\Omega_h} + \|\mathbf{g}_j(\tau)\|^2_{\partial\Omega_h})d\tau\right)$$

Initial observation

- Well-posedness and Stability are similar concepts.
  - Energy estimates required in both.
  - Well-posedness additionally demand: uniqueness and existence.
- Should be possible to develop most of the theory on the continuous side.
  - Easier to work with the continuous problems.
  - When done, generalize to the discrete case.

Nonlinear vs linear theory

- The linear theory is <u>complete</u>.
  - An energy estimate bounds the solution.
  - Uniqueness and error estimates follows.
  - Existence is given by using a minimal number of boundary conditions.
- The theory for ”almost linear” (smooth) nonlinear problems is <u>complete</u>.
  - The linearization and localisation principles
- The fully nonlinear theory is <u>incomplete</u>.
  - Energy (entropy) estimates bounds (maybe) the solution.
  - Uniqueness, error estimates and existence are generally not known.

Motivation and strategy

- Well-posedness of the continuous problem is a fundamental requirement in numerical calculations (otherwise, convergence to what?).
- Well-posedness depends almost only on the boundary/interface conditions.
- Discretization techniques on SBP- SAT form (FD, FEM, SEM, DG, FR) add technical difficulties, not fundamental ones.
- We focus on the PDE + boundary conditions and derive boundary procedures that lead to a well posed continuous problem.
- Energy stability for the discrete approximation will follow almost automatically.

The roadmap

1. ***The symmetrization***: The energy method requires symmetric matrices such that Integration-By-Parts (IBP) can be performed.
2. ***The continuous energy method:*** By multiplying with the solution, integrating over the domain and using IBP, the energy rate involving an indefinite boundary term is obtained.
3. ***The number of boundary conditions***: The number of boundary conditions is equal to the number of eigenvalues with negative sign in the boundary term.
4. ***The form of the boundary conditions***: The variables that correspond to negative eigenvalues are specified in terms of the corresponding positive ones and data.
5. ***The weak implementation***: The boundary conditions are imposed using penalty terms such that the boundary term becomes negative semi-definite for zero boundary data.
6. ***The discrete approximation***: The problem is discretized using Summation-By-Parts (SBP) operators and penalty terms from the continuous problem.
7. ***The discrete energy method:*** Stability is proved by using the energy method and making sure that the discrete energy rate mimics the continuous one.

1. The symmetrization

Take a non-symmetric system of equations

$$V_t + \tilde{A}V_x + \tilde{B}V_y + \tilde{C}V_z = \tilde{F}_x + \tilde{G}_y + \tilde{H}_z$$

Multiply with a symmetrizer S such that a symmetrized system is obtained:

$$U_t + \bar{A}U_x + \bar{B}U_y + \bar{C}U_z = \bar{F}_x + \bar{G}_y + \bar{H}_z$$

Choose S such that:

$$U = S^{-1}V,\ \bar{A} = S^{-1}\tilde{A}S,\ \bar{B} = S^{-1}\tilde{B}S,\ \bar{C} = S^{-1}\tilde{C}S \text{ and } \bar{D}_{ij} = S^{-1}\tilde{D}_{ij}S.$$

For more details, see: Abarbanel & Gottlieb JCP 1981

The prototype problem

$$
\begin{aligned}
U_t + \bar{A}U_x + \bar{B}U_y + \bar{C}U_z &= \bar{F}_x + \bar{G}_y + \bar{H}_z, \quad (x,y,z) \in \Omega, \quad t \geq 0 \\
HU &= g, \quad (x,y,z) \in \delta\Omega, \quad t \geq 0 \\
U &= f, \quad (x,y,z) \in \Omega, \quad t = 0.
\end{aligned}
$$

$$
\begin{aligned}
F &= D_{11}U_x + D_{12}U_y + D_{13}U_z \\
G &= D_{21}U_x + D_{22}U_y + D_{23}U_z \\
H &= D_{31}U_x + D_{32}U_y + D_{33}U_z
\end{aligned}
\qquad
D_{ij} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & X & X & X & X \\ 0 & X & X & X & X \\ 0 & X & X & X & X \\ 0 & X & X & X & X \end{bmatrix}
\qquad
F, G, H = \begin{bmatrix} 0 \\ X \\ X \\ X \\ X \end{bmatrix}
$$

An <u>incompletely</u> parabolic system of equations.

2. The continuous energy method

$$||U||_t^2 + 2DI_c = BT$$

$$DI_c = \int_\Omega \begin{bmatrix} U_x \\ U_y \\ U_z \end{bmatrix}^T \begin{bmatrix} \bar{D}_{11} & \bar{D}_{12} & \bar{D}_{13} \\ \bar{D}_{21} & \bar{D}_{22} & \bar{D}_{23} \\ \bar{D}_{31} & \bar{D}_{32} & \bar{D}_{33} \end{bmatrix} \begin{bmatrix} U_x \\ U_y \\ U_z \end{bmatrix} dx\,dy\,dz > 0$$

$$BT = -\oint_{\partial\Omega} U^T A U - 2U^T F ds$$

How do we choose the boundary operator <u>H to bound BT ?</u>

LiU

Blocking it up

$$BT = -\oint_{\partial\Omega} U^T A U - 2U^T F ds$$

$$\longrightarrow \quad A = n_1\tilde{A} + n_2\tilde{B} + n_3\tilde{C}, \quad F = n_1\bar{F} + n_2\bar{G} + n_3\bar{H} \quad \longleftarrow$$

$$U = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}, \quad F = \begin{bmatrix} 0 \\ F_2 \end{bmatrix}, \quad A = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{bmatrix} \quad \Longrightarrow$$

$$U^T A U - 2U^T F = \begin{bmatrix} U_1 \\ U_2 \\ F_2 \end{bmatrix}^T \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12}^T & A_{22} & -I \\ 0 & -I & 0 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ F_2 \end{bmatrix}$$

Indefinite

A rotation

$$\begin{bmatrix} U_1 \\ U_2 \\ F_2 \end{bmatrix}^T \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12}^T & A_{22} & -I \\ 0 & -I & 0 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ F_2 \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}^T R^T \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12}^T & A_{22} & -I \\ 0 & -I & 0 \end{bmatrix} R \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

$$= \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}^T \begin{bmatrix} A_{11} & 0 & 0 \\ 0 & \tilde{A}_{22} & 0 \\ 0 & 0 & -(\tilde{A}_{22})^{-1} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = R^{-1} \begin{bmatrix} U_1 \\ U_2 \\ F_2 \end{bmatrix} = \begin{bmatrix} U_1 + (A_{11})^{-1} A_{12} U_2 \\ U_2 - (\tilde{A}_{22})^{-1} F_2 \\ F_2 \end{bmatrix}$$

Block-diagonal to diagonal

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}^T \begin{bmatrix} A_{11} & 0 & 0 \\ 0 & \tilde{A}_{22} & 0 \\ 0 & 0 & -(\tilde{A}_{22})^{-1} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

$$\tilde{A}_{22} = A_{22} - A_{12}^T (A_{11})^{-1} A_{12} \qquad \tilde{A}_{22} = \tilde{A}_{22}^T \implies$$

$$\tilde{A}_{22} = X \Lambda_{22} X^T \text{ where } \Lambda_{22} = diag(\Lambda_{22}^+, \Lambda_{22}^-) \text{ and } X = [X_+, X_-]$$

$$BT = -\oint_{\delta\Omega} \begin{bmatrix} w_1 \\ X_+^T w_2 \\ X_-^T w_2 \\ X_+^T w_3 \\ X_-^T w_3 \end{bmatrix}^T \begin{bmatrix} A_{11} & 0 & 0 & 0 & 0 \\ 0 & \Lambda_{22}^+ & 0 & 0 & 0 \\ 0 & 0 & \Lambda_{22}^- & 0 & 0 \\ 0 & 0 & 0 & -(\Lambda_{22}^+)^{-1} & 0 \\ 0 & 0 & 0 & 0 & -(\Lambda_{22}^-)^{-1} \end{bmatrix} \begin{bmatrix} w_1 \\ X_+^T w_2 \\ X_-^T w_2 \\ X_+^T w_3 \\ X_-^T w_3 \end{bmatrix} ds$$

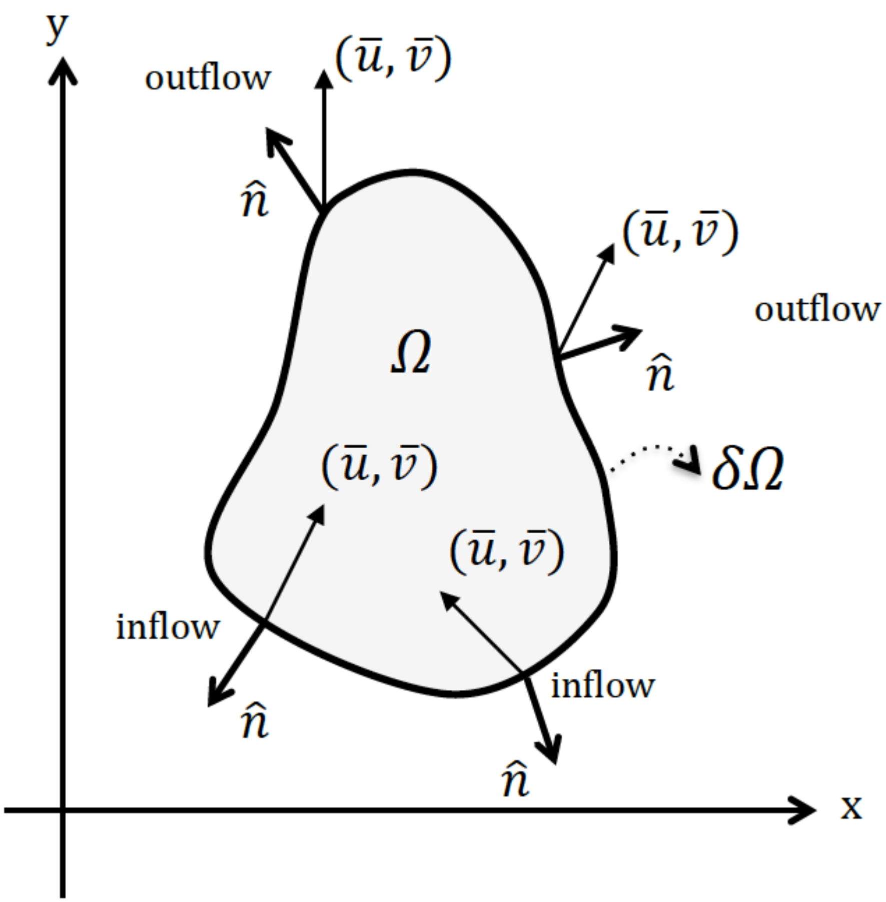
3. The number of boundary conditions

$$BT = -\oint_{\delta\Omega} \begin{bmatrix} w_1 \\ X_+^T w_2 \\ X_-^T w_2 \\ X_+^T w_3 \\ X_-^T w_3 \end{bmatrix}^T \begin{bmatrix} A_{11} & 0 & 0 & 0 & 0 \\ 0 & \Lambda_{22}^+ & 0 & 0 & 0 \\ 0 & 0 & \Lambda_{22}^- & 0 & 0 \\ 0 & 0 & 0 & -(\Lambda_{22}^+)^{-1} & 0 \\ 0 & 0 & 0 & 0 & -(\Lambda_{22}^-)^{-1} \end{bmatrix} \begin{bmatrix} w_1 \\ X_+^T w_2 \\ X_-^T w_2 \\ X_+^T w_3 \\ X_-^T w_3 \end{bmatrix} ds$$

- The number of boundary conditions is equal to the number of negative entries in $A_{11}, \Lambda_{22}^+ \text{ and } -(\Lambda_{22}^-)^{-1}$
- That number varies only with A_{11} since the the total number of entries in $\Lambda_{22}^+ \text{ and } -(\Lambda_{22}^-)^{-1}$ is constant and equal to the number of eigenvalues in $\tilde{A}_{22}$.
- <u>The number depends on the original matrices in the problem.</u>

The Navier-Stokes and Euler equations

- In the Navier-Stokes equations, $A_{11} = (\bar{u}, \bar{v}, \bar{w}) \cdot \hat{n}$ and the number of boundary conditions depends only on the direction of the flow,
- The fact that the number of boundary conditions is independent of the speed (sub or supersonic) of the flow is quite different from the case for the Euler equations.
- In the limit of infinite Reynolds numbers we get the number of boundary conditions for the Euler equations.



4. The form of the boundary conditions

$$
\begin{aligned}
BT = & - \oint_{\delta\Omega} \begin{bmatrix} 1_{+}(\gamma^{+})w_1 \\ X_{+}^{T} w_2 \\ X_{-}^{T} w_3 \end{bmatrix}^{T} \begin{bmatrix} \gamma^{+} & 0 & 0 \\ 0 & \Lambda_{22}^{+} & 0 \\ 0 & 0 & -(\Lambda_{22}^{-})^{-1} \end{bmatrix} \begin{bmatrix} 1_{+}(\gamma^{+})w_1 \\ X_{+}^{T} w_2 \\ X_{-}^{T} w_3 \end{bmatrix} ds \\
& - \oint_{\delta\Omega} \begin{bmatrix} 1_{-}(\gamma^{-})w_1 \\ X_{-}^{T} w_2 \\ X_{+}^{T} w_3 \end{bmatrix}^{T} \begin{bmatrix} \gamma^{-} & 0 & 0 \\ 0 & \Lambda_{22}^{-} & 0 \\ 0 & 0 & -(\Lambda_{22}^{+})^{-1} \end{bmatrix} \begin{bmatrix} 1_{-}(\gamma^{-})w_1 \\ X_{-}^{T} w_2 \\ X_{+}^{T} w_3 \end{bmatrix} ds \\
= & - \oint_{\delta\Omega} \begin{bmatrix} W^{+} \\ W^{-} \end{bmatrix}^{T} \begin{bmatrix} \Lambda^{+} & 0 \\ 0 & \Lambda^{-} \end{bmatrix} \begin{bmatrix} W^{+} \\ W^{-} \end{bmatrix} ds
\end{aligned}
$$

Specify the variables that can cause growth

$$\|U\|_t^2 + 2DI_c = -\oint_{\delta\Omega} \begin{bmatrix} W^+ \\ W^- \end{bmatrix}^T \begin{bmatrix} \Lambda^+ & 0 \\ 0 & \Lambda^- \end{bmatrix} \begin{bmatrix} W^+ \\ W^- \end{bmatrix} ds.$$

Proposition 1. *The general form of the boundary condition that bound BT and lead to well-posedness is*

$$W^- = RW^+ + g$$

R is a matrix with the number of rows equal to the number of boundary conditions and g is given boundary data.

In terms of the original formulation, we have:

$$HU = (H^- - RH^+)U, \; H^-U = W^-, \; H^+U = W^+$$

Remark: <u>The boundary operator H depends on the original matrices in the problem.</u>

Strong homogeneous boundary conditions

$$\|U\|_t^2 + 2DI_c = -\oint_{\delta\Omega} (W^+)^T (R^T \Lambda^- R + \Lambda^+)(W^+) ds$$

Proof of Proposition 1: *The right-hand-side is bounded if R such that*

$$R^T \Lambda^- R + \Lambda^+ \geq 0$$

Remark: *For strongly imposed boundary conditions, some variables are replaced. Here for example only* W^+ *is present.*

5. The weak implementation

$$\|U\|_t^2 + 2DI_c = -\oint_{\delta\Omega} \begin{bmatrix} W^+ \\ W^- \end{bmatrix}^T \begin{bmatrix} \Lambda^+ & 0 \\ 0 & \Lambda^- \end{bmatrix} \begin{bmatrix} W^+ \\ W^- \end{bmatrix}$$

$$+ U^T \Sigma (W^- - RW^+) + (U^T \Sigma (W^- - RW^+))^T \, ds$$

Introduce Σ *such that* $\Sigma = (H^-)^T \Lambda^-$ $\Longrightarrow$

$$\|U\|_t^2 + 2DI_c = -\oint_{\delta\Omega} (W^+)^T (R^T \Lambda^- R + \Lambda^+)(W^+) \, ds$$

$$+ \oint_{\delta\Omega} (W^- - RW^+)^T \Lambda^- (W^- - RW^+) \, ds$$

Proof of Proposition 1. *The right-hand-side is bounded.*

Weak homogeneous boundary conditions

$$\|U\|_t^2 + 2DI_c = -\oint_{\delta\Omega} (W^+)^T (R^T \Lambda^- R + \Lambda^+)(W^+) + \oint_{\delta\Omega} (W^- - RW^+)^T \Lambda^- (W^- - RW^+)\, ds$$

Remark: The weak imposition of boundary conditions produces the strong energy estimate with an additional damping term.

Remark: For weakly imposed boundary conditions, all variables are kept and present in the energy estimate.

Strong non-homogeneous boundary conditions

$$\|U\|_t^2 + 2DI_c = -\oint_{\delta\Omega} \begin{bmatrix} W^+ \\ g \end{bmatrix}^T \begin{bmatrix} R^T\Lambda^- R + \Lambda^+ & R^T\Lambda^- \\ \Lambda^- R & \Lambda^- \end{bmatrix} \begin{bmatrix} W^+ \\ g \end{bmatrix} ds$$

Add and subtract $g^T G g$ where G is a positive definite bounded matrix

$$\|U\|_t^2 + 2DI_c = -\oint_{\delta\Omega} \begin{bmatrix} W^+ \\ g \end{bmatrix}^T \begin{bmatrix} R^T\Lambda^- R + \Lambda^+ & R^T\Lambda^- \\ \Lambda^- R & G \end{bmatrix} \begin{bmatrix} W^+ \\ g \end{bmatrix} ds + \oint_{\delta\Omega} g^T (G + |\Lambda^-|) g \, ds.$$

We have an estimate in terms of data if:

$$G \geq (\Lambda^- R)(R^T\Lambda^- R + \Lambda^+)^{-1}(\Lambda^- R)^T \qquad R^T\Lambda^- R + \Lambda^+ > 0$$

Weak non-homogeneous boundary conditions

$$\|U\|_t^2 + 2DI_c = -\oint_{\delta\Omega} \begin{bmatrix} W^+ \\ W^- \end{bmatrix}^T \begin{bmatrix} \Lambda^+ & 0 \\ 0 & \Lambda^- \end{bmatrix} \begin{bmatrix} W^+ \\ W^- \end{bmatrix} ds$$
$$+ \oint_{\delta\Omega} U^T \Sigma (W^- - RW^+ - g) + (U^T \Sigma (W^- - RW^+ - g))^T ds$$

Introduce $\Sigma = (H^-)^T \Lambda^-$ $\Longrightarrow$

$$\|U\|_t^2 + 2DI_c = -\oint_{\delta\Omega} \begin{bmatrix} W^+ \\ W^- \\ g \end{bmatrix}^T \underbrace{\begin{bmatrix} \Lambda^+ & R^T \Lambda^- & 0 \\ \Lambda^- R & -\Lambda^- & \Lambda^- \\ 0 & \Lambda^- & 0 \end{bmatrix}}_{M} \begin{bmatrix} W^+ \\ W^- \\ g \end{bmatrix} ds$$

Weak non-homogeneous boundary conditions

$$M = \begin{bmatrix} -R^T\Lambda^- R & R^T\Lambda^- & -R^T\Lambda^- \\ \Lambda^- R & -\Lambda^- & \Lambda^- \\ -\Lambda^- R & \Lambda^- & -\Lambda^- \end{bmatrix} + \begin{bmatrix} R^T\Lambda^- R + \Lambda^+ & 0 & R^T\Lambda^- \\ 0 & 0 & 0 \\ \Lambda^- R & 0 & G \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -G + \Lambda^- \end{bmatrix}$$

The second and third matrices OK from strong analysis. Must make sure that the first matrix is positive semi-definite.

$$\begin{bmatrix} -R^T\Lambda^- R & R^T\Lambda^- & -R^T\Lambda^- \\ \Lambda^- R & -\Lambda^- & \Lambda^- \\ -\Lambda^- R & \Lambda^- & -\Lambda^- \end{bmatrix} = \begin{bmatrix} R & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}^T (C_0 \otimes \Lambda^-) \begin{bmatrix} R & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}, \quad C_0 = \begin{bmatrix} -1 & +1 & -1 \\ +1 & -1 & +1 \\ -1 & +1 & -1 \end{bmatrix}$$

C_0 has eigenvalues (-3,0,0) and hence the first matrix is positive semi-definite.

Weak non-homogeneous boundary conditions

The rest term due to the first matrix (the deviation from the strong case) is:

$$
\begin{aligned}
\tilde{R} &= -\oint_{\delta\Omega} \begin{bmatrix} W^+ \\ W^- \\ g \end{bmatrix}^T \begin{bmatrix} R & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}^T (C_0 \otimes \Lambda^-) \begin{bmatrix} R & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} W^+ \\ W^- \\ g \end{bmatrix} ds \\
&= -\oint_{\delta\Omega} \begin{bmatrix} RW^+ \\ W^- \\ g \end{bmatrix}^T (X\Gamma X^T \otimes \Lambda^-) \begin{bmatrix} RW^+ \\ W^- \\ g \end{bmatrix} ds \\
&= -\oint_{\delta\Omega} \begin{bmatrix} W^- - RW^+ - g \\ RW^+ + W^- \\ RW^+ - W^- + 2g \end{bmatrix}^T \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \otimes \Lambda^- \begin{bmatrix} W^- - RW^+ - g \\ RW^+ + W^- \\ RW^+ - W^- + 2g \end{bmatrix} ds \\
&= +\oint_{\delta\Omega} (W^- - RW^+ - g)^T \Lambda^- (W^- - RW^+ - g)\, ds \le 0.
\end{aligned}
$$

Strong and weak non-homogeneous boundary conditions

The results for the homogeneous cases generalize to the non-homogenous cases. However, we must strengthen the condition on R to

$$R^T \Lambda^- R + \Lambda^+ > 0$$

Remark: The weak imposition of boundary conditions produces the strong energy estimate with an additional damping term.

Differences and similarities between weak and strong boundary conditions

- For strongly imposed boundary conditions, some variables are replaced in the energy estimate.
- For weakly imposed boundary conditions, all variables are present in the energy estimate.
- Weak boundary conditions produce strong energy estimates with an additional damping term.

6. The discrete approximation

$$
\begin{aligned}
V_t &+ (D_x \otimes I_y \otimes I_z \otimes A)V + (I_x \otimes D_y \otimes I_z \otimes B)V + (I_x \otimes I_y \otimes D_z \otimes C)V \\
&= (D_x \otimes I_y \otimes I_z \otimes I_M)F + (I_x \otimes D_y \otimes I_z \otimes I_M)G + (I_x \otimes I_y \otimes D_z \otimes I_M)H \\
&+ \underline{(E_N P_x^{-1}\Sigma \otimes I_y \otimes I_z \otimes I_M)((\tilde{H}^+ - \tilde{R}\tilde{H}^-)V - e_{N_x} \otimes g)} \\
V(0) &= f.
\end{aligned}
$$

$$
\begin{aligned}
\tilde{F} &= (\tilde{I} \otimes \bar{D}_{11})V_x + (\tilde{I} \otimes \bar{D}_{12})V_y + (\tilde{I} \otimes \bar{D}_{13})V_z \\
\tilde{G} &= (\tilde{I} \otimes \bar{D}_{21})V_x + (\tilde{I} \otimes \bar{D}_{22})V_y + (\tilde{I} \otimes \bar{D}_{23})V_z \\
\tilde{H} &= (\tilde{I} \otimes \bar{D}_{31})V_x + (\tilde{I} \otimes \bar{D}_{32})V_y + (\tilde{I} \otimes \bar{D}_{33})V_z,
\end{aligned}
$$

$$\tilde{I} = (I_x \otimes I_y \otimes I_z)$$

$$
\begin{aligned}
V_x &= (D_x \otimes I_y \otimes I_z \otimes I_M)V, \\
V_y &= (I_y \otimes D_y \otimes I_z \otimes I_M)V, \\
V_z &= (I_z \otimes I_y \otimes D_z \otimes I_M)V.
\end{aligned}
$$

With the weak penalty term at x=1, i=N.

7. The discrete energy method

$$\frac{d}{dt}\|V\|^2_{P_{xyz}} + 2DI_d = - \begin{bmatrix}\tilde{W}^+ \\ \tilde{W}^-\end{bmatrix}^T_N \left(P_{yz} \otimes \begin{bmatrix}\Lambda^+ & 0 \\ 0 & \Lambda^-\end{bmatrix}\right) \begin{bmatrix}\tilde{W}^+ \\ \tilde{W}^-\end{bmatrix}_N + V^T\tilde{\Sigma}(E_N \otimes P_y \otimes P_z \otimes I_M)(\tilde{W}^- - \tilde{R}\tilde{W}^+) + (\tilde{W}^- - \tilde{R}\tilde{W}^+)^T(E_N \otimes P_y \otimes P_z \otimes I_M)\tilde{\Sigma}^T V$$

$$\|V\|^2_{P_{xyz}} = V^T(P_x \otimes P_y \otimes P_z \otimes I_M)V$$

$$DI_d = \begin{bmatrix}V_x \\ V_y \\ V_z\end{bmatrix}^T P_{xyz} \begin{bmatrix}\tilde{I}\otimes\bar{D}_{11} & \tilde{I}\otimes\bar{D}_{12} & \tilde{I}\otimes\bar{D}_{13} \\ \tilde{I}\otimes\bar{D}_{21} & \tilde{I}\otimes\bar{D}_{22} & \tilde{I}\otimes\bar{D}_{23} \\ \tilde{I}\otimes\bar{D}_{31} & \tilde{I}\otimes\bar{D}_{32} & \tilde{I}\otimes\bar{D}_{33}\end{bmatrix} \begin{bmatrix}V_x \\ V_y \\ V_z\end{bmatrix} > 0$$

Weak homogeneous boundary conditions

The ”continuous” choice $\Sigma = (H^-)^T(I \otimes \Lambda^-) \quad \Longrightarrow$

$$\frac{d}{dt}\|V\|^2_{P_{xyz}} + 2DI_d = -(\tilde{W}_N^+)^T(P_{yz} \otimes (R^T\Lambda^- R + \Lambda^+))(\tilde{W}_N^+) + (\tilde{W}_N^- - R\tilde{W}_N^+)^T(P_{yz} \otimes \Lambda^-)(\tilde{W}_N^- - R\tilde{W}_N^+)$$

Compare with the continuous estimate.

$$\|U\|_t^2 + 2DI_c = -\oint_{\delta\Omega}(W^+)^T(R^T\Lambda^- R + \Lambda^+)(W^+)\,ds + \oint_{\delta\Omega}(W^- - RW^+)^T\Lambda^-(W^- - RW^+)\,ds$$

Remark: *The same result as in the continuous case, with an* *<u>additional term that adds a small amount of dissipation</u>*.

Weak non-homogeneous boundary conditions

All the results generalize: $\Sigma = (H^-)^T (I \otimes \Lambda^-) \implies$

Proposition 2. *The semi-discrete approximation with weak homogeneous or non-homogeneous boundary conditions and <u>the same penalty matrix as in the continuous case, is stable or strongly stable.</u>*

Remark: *The discrete estimates mimic the continuous ones term by term, and the approximations are stable. <u>This implies strict stability, i.e. the discrete energy grows or decays with the same rate as the continuous one.</u>*

Remark:. *<u>The derivation in the weak cases is completely analogous to the continuous derivation.</u> Both the boundary conditions and the penalty matrix are already derived.*

Summary and conclusions

- We have provided ”A Roadmap for Well Posed and Stable Problems in Computational Physics”.
- It is valid for most (if not all) problems in computational physics.
- The general boundary conditions lead to strongly well posed problems both for the weak and strong imposition.
- The well-posedness analysis lead directly to energy-stability of numerical approximations on SBP-SAT form.
- <u>The key to a good numerical method is knowledge of well-posedness of the PDE!</u>

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