MAI0126 TOPOLOGICAL COMBINATORICS, 2014
PROBLEM SET 1/2

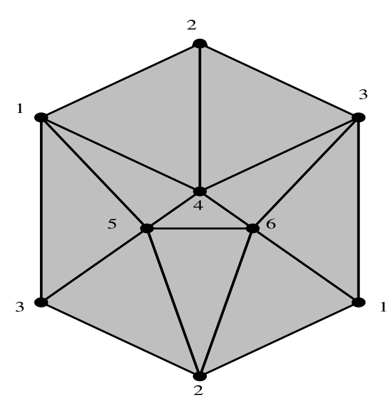
UPDATED 2014-10-17; FINAL VERSION

Each problem is worth five points. You may get partial credit for non-useless, non-perfect solutions. The problems are not ordered by difficulty. Please hand in your solutions no later than **November 6**.

1.1. Consider the statement "*Suppose* $f : B^n \to \mathbb{R}^n$ *is continuous and antipodal on the boundary* ∂B^n*. Then,* $f(x) = 0$ *for some* $x \in B^n$*.*" Prove that it is equivalent to the Borsuk-Ulam theorem.

1.2. Consider the statement "*Suppose* $f : S^n \to S^{n-1}$ *is continuous. Then, there exists some* $x \in S^n$ *such that* $f(-x) = f(x)$*.*" Prove that it is equivalent to the Borsuk-Ulam theorem.

2.1. Below is a triangulation of the real projective plane $\mathbb{RP}_2$ with 10 triangles, 15 edges and 6 vertices. (Notice the identification of vertices and edges on the "outer rim".) Use this triangulation to compute all the reduced homology groups $\widetilde{H}_i(\mathbb{RP}_2; k)$, where k is a field.
Beware: the answer is dependent on the characteristic of k*.*



2.2. Suppose Δ is a simplicial complex where all facets have dimension n. Assume furthermore that every $(n-1)$-dimensional face of Δ is contained in exactly two facets. Show that $\widetilde{H}_n(\Delta; \mathbb{F}_2)$ is nontrivial, where $\mathbb{F}_2$ is the field with two elements.

3.1. Suppose $G = (V, E)$ is a finite graph which has an automorphism of order two with no fixed vertices or edges. Show that the number of odd-degree vertices of

G is divisible by four. Deduce that the boundary of a 1-dimensional, antipodally symmetric element of the chain group $C_1(\diamond^n; \mathbb{F}_2)$ cannot be the sum of two points.
Hint: Use the handshaking lemma.
Remark: This completes our proof of Tucker's lemma.

3.2. Show the generalised version of Lyusternik-Shnirel'man. That is, suppose $X_1, \ldots, X_{n+1}$ form a covering of S^n with each X_i being open or closed. Prove that for some $i \in [n+1]$, X_i contains a pair of antipodal points.
Hint: Induct on the number of closed sets. For the induction step, replace a closed set with a suitably constructed open one.

4.1. Show that the box complex of the complete graph K_n is isomorphic (as a simplicial complex) to the complex obtained from $\diamond^{n-1}$ by removing two antipodal facets.

5.1. Prove that no graph has a contractible neighbourhood complex.

5.2. Show that the neighbourhood complex of the complete graph K_n is $(n-3)$-connected but not $(n-2)$-connected.

6.1 Suppose G is a connected, bipartite graph. Prove that the neighbourhood complex of G has two connected components that are homotopy equivalent to each other.
Hint: Show that $L(G)$ has two components that are interchanged by the isomorphism CN.