MAI0126 TOPOLOGICAL COMBINATORICS, 2014
PROBLEM SET 2/2

UPDATED 2014-11-27; FINAL VERSION

Each problem is worth five points. You may get partial credit for non-useless, non-perfect solutions. The problems are not ordered by difficulty. Please hand in your solutions no later than **December 15**.

7.1. Let $n \geq 3$ be an integer. Consider the simplicial complex Δ_n which consists of the graphs on vertex set $[n]$ such that the connected component which contains 1 has at most $n-2$ vertices.[1] Prove that Δ_n is collapsible.

8.1. Given a graph $G = (V, E)$, the *independence complex* $\mathrm{Ind}(G)$ is the simplicial complex whose simplices are the independent subsets of V.[2] For a vertex v, let $N(v) = \{w \in V \mid \{v, w\} \in E\}$. Suppose that $N(v_1) \supseteq N(v_2)$ for distinct vertices $v_1, v_2 \in V$. Prove that $\mathrm{Ind}(G)$ is homotopy equivalent to $\mathrm{Ind}(G - v_1)$, where $G - v_1$ is the graph obtained from G by removing the vertex v_1 and all edges containing it.
Hint. Define an acyclic matching on $\mathrm{Ind}(G)$ *with* $\mathrm{Ind}(G \setminus v_1)$ *as set of critical faces.*

8.2. Construct a simplicial complex which is nonevasive but not a cone.

9.1. Let $\Delta_n^{\leq k}$ denote the k-skeleton of the n-dimensional simplex. That is, the simplices of $\Delta_n^{\leq k}$ are the subsets of $[n+1]$ that contain at most $k+1$ elements. Use discrete Morse theory for proving that $\Delta_n^{\leq k}$ is homotopy equivalent to a wedge of $\binom{n}{k+1}$ spheres of dimension k.

10.1. With notation as in Problem 9.1, construct a shelling of $\Delta_n^{\leq k}$ and prove directly[3] that that there are exactly $\binom{n}{k+1}$ homology facets.

11.1. For a positive integer n, recall that D_n is the poset whose elements are the positive divisors of n partially ordered by declaring $x \leq y$ iff x divides y. Clearly, n is the maximum and 1 is the minimum of D_n. Prove that D_n has an EL-labelling. What is the homotopy type of $\Delta(D_n \setminus \{1, n\})$?

---

[1] By this we mean that the vertex set of Δ_n is the set of pairs $\{i, j\}$, $1 \leq i < j \leq n$, and σ is a simplex in Δ_n iff the graph with vertex set $[n]$ and edge set σ has the described property. (Note that the property is preserved under removal of edges, so Δ_n is indeed a simplicial complex.)

[2] Recall that a set of vertices of a graph is independent if no pair of vertices in the set is connected by an edge.

[3] That is, without referring to the known (from Problem 9.1 or elsewhere) homotopy type of the $\Delta_n^{\leq k}$.

12.1. Let $\mathcal{V}$ be a vector space and $S \subseteq \mathcal{V}$ a finite set of vectors that span $\mathcal{V}$. A *flat* is a subspace of the form $\mathcal{U} = \operatorname{span} T$ for some $T \subseteq S$. Prove that the set of flats partially ordered by inclusion form a lattice L.

12.2. With notation as in Problem 12.1, define $\Delta(S) = \{\sigma \subset S \mid \operatorname{span} \sigma \neq \mathcal{V}\}$. Observe that $\Delta(S)$ is a simplicial complex. Show that $\Delta(S)$ and $\Delta(L \setminus \{\{0\}, \mathcal{V}\})$ are homotopy equivalent.