Exercises MAI0128 Introduction to Algebraic Geometry. **I** Planar Projective Conics and Cubics

Exercise 1 *Show that the subset $\mathcal{C}$ of $\mathbb{C}^2$ consisting of the points of the form $(x, y) = (t^2, t^3 + 1), t \in \mathbb{C}$ is a complex algebraic curve. Give a parametrizacion of corresponding projective curve. Show that any polynomial $h(x; y)$ vanishing at $\mathcal{C}$ is divisible by $f(x, y) = (y - 1)^2 - x^3$. Calculate the singular points of it. Which curve is $\mathcal{C}$?*

Exercise 2 *Find singular points, their multiplicity and the tangent lines to the curves at the singular points of the projective curves:*

1. $x^3y + y^3z + z^3x = 0$
2. $x^2y^3 + y^2z^3 + z^2x^3 = 0$
3. *Fermat's curves* $x^n + y^n - z^n = 0, n \geq 3$.
4. *Generalized Fermat's curve of degree 3* $x^3 + y^3 + z^3 + 3\lambda(xyz) = 0, \lambda \in \mathcal{C}$

Exercise 3 *Show that a singular point $p(a, b)$ of an affine curve $\mathcal{C}$ in $\mathbb{C}^2$ defined by the polynomial $f(x, y)$ is an ordinary double point iff*
$(\frac{\partial^2 f(a,b)}{\partial x \partial y})^2 \neq (\frac{\partial^2 f(a,b)}{\partial x^2})(\frac{\partial^2 f(a,b)}{\partial y^2})$.

Exercise 4 *Give examples showing that all possible ways two projective conics intersect are realizable. In all the examples one of the intersection point is $[0 : r_1 : r_2]$, where r_1, r_2 are the two last digits in the year you were born (the coordinates are $[x_0 : x_1 : x_2]$).*

Exercise 5 *Consider the planes $\mathcal{H}_1 = V(x_0 - 1)$ (inner coordinates $[(x, y))$ and $\mathcal{H}_2 = V(x_1 - 1)$ (inner coordinates $(z', y']$) in $\mathbb{R}^3$. Consider the dominant map $\varphi : \mathcal{H}_1 - \to \mathcal{H}_2$ given by $\varphi((x, y)) = (z' = 1/x, y' = y/x)$. Which are the points were neither φ nor φ^{-1} are not defined, and their images in $\mathbb{P}^2_{\mathbb{R}}$? Give the image of the line $r_1 x = y + r_2$ and the circles $(x_1)^2 + (y)^2 = r_2, (x_1)^2 + (y)^2 = 1/r_2$, where r_1, r_2 are as in the previous exercise.*

Exercise 6 *Use Bezout's theorem to show that if a projective curve $\mathcal{C}$ in $\mathbb{P}^2$ of degree d has strictly more than $d/2$ singular points all lying in a line L then L is a component of $\mathcal{C}$.*

Exercise 7 *Consider the cubic defined by the polynomial $x_2^2 x_0 - x_1^3 - r_1 x_1 x_0^2 - r_2 x_0^3$. Calculate the inflection points. is there any inflection point O at infinity? In that case, determine the points on the curve of order 2 under the addition having O as zero. Calculate the points of order 4. r_1, r_2 are as in previous exercises.*

Exercise 8 *Consider the nonsingular cubic* $\mathcal{C}$ *in* $\mathbb{P}^2$ *defined by* $x_1^3 + x_2^3 + x_0^3 + 3r_2(x_1x_2x_3) = 0$ *(see exercise 1). Show that the points of inflection of* $\mathcal{C}$ *are the points of intersection of* $\mathcal{C}$ *with a different curve of the same form. Thus* $\mathcal{C}$ *has 9 inflection points. Show that they satisfy:* $x_1^3 + x_2^3 + x_0^3 = (x_1x_2x_3) = 0$.

Thus one can show that

$0 = (x_1 + x_2 + x_0)(x_1 + wx_2 + w^2x_0)(x_1 + w^2x_2 + wx_0)$

$0 = (x_1 + wx_2 + wx_0)(x_1 + w^2x_2 + x_0)(x_1 + x_2 + w^2x_0)$

$0 = (x_1+w^2x_2+w^2x_0)(x_1+wx_2+x_0)(x_1+wx_2+wx_0)$, *where* $w = e^{2\pi i/3}$. *Deduce that a line through any of two inflection points meets* $\mathcal{C}$ *in a third inflection point.*

Exercise 9 *Consider the nonsingular cubic* $\mathcal{C}$ *defined by the polynomial* $x_2^2x_0 - x_1(x_1 - x_0)(x_1 - \lambda x_0), \lambda \neq 0, 1$. *Show that given an inflection point* $O \in \mathcal{C}$, *then there are exactly four tangent lines to* $\mathcal{C}$ *passing through* O.

Exercise 10 *Consider the affine spatial curve* $\mathcal{C}$ *parametrized by* $(1+\cos(u), \sin(u), \sin(u/2))$, $u \in (0, 2\pi)$. *Show that the curve is the intersection of a sphere* $x^2 + y^2 + z^2 = 4$ *and a cylinder* $(x-1)^2 + y^2 = 1$. *Show that the function* $g : (0, \infty) \to (0, 2\pi)$ *given by* $g(t) = 4arctang(t)$ *gives a rational parametrization of* $\mathcal{C}$. *Determine such parametrization.*