Exercises MAI0128 Introduction to Algebraic Geometry. **2** Affine Varieties and Their Functions.

Exercise 1 *Let* $J = \langle xy, xz, yz\rangle$ *in* $k[x,y,z]$. *Find* $\mathbf{V}(J)$ *and* $\mathbf{I}(\mathbf{V}(J))$ *in* k^3. *Is* $\mathbf{V}(J)$ *irreducible?. Show that* J *cannot be generated by 2 polynomials.*

Consider $J' = \langle xy, (x-y)z\rangle$. *Calculate* $V(J')$ *and* $\sqrt{J'}$.

Calculate $V(J')$ *and* $\sqrt{J'}$ *for* $J' = \langle x^2+y^2+z^2, xy+xz+yz\rangle$.

Exercise 2 *Show that the irreducible components in an irredundant decomposition of an algebraic variety (set in Reid's notation) are unique*

Exercise 3 *Identify* $V(J) = \mathbf{V}\langle x^2-yz, y^4-xz, z^2-y^3x\rangle$. *Identify* $\mathbf{V}\langle x^2-yz, y^4-xz\rangle$, $\mathbf{V}\langle x^2-yz, z^2-y^3x\rangle$ *and* $\mathbf{V}\langle y^4-xz, z^2-y^3x\rangle$.

Exercise 4 *Let* $f \in \mathbb{C}[x_1,\ldots,x_n]$ *and let* $f = f_1^{a_1} f_2^{a_2} \ldots f_r^{a_r}$ *be the decomposition of* f *into irreducible factors. Show that* $\mathbf{V}(f) = \mathbf{V}(f_1) \cup \cdots \cup \mathbf{V}(f_r)$ *is the decomposition of* $\mathbf{V}(f)$ *into irreducible components and* $\mathbf{I}(\mathbf{V}(f)) = \langle f_1 f_2 \ldots f_r\rangle$.

Exercise 5

1. *Show that the intersection of any collection of prime ideals is radical.*
2. *Show that an irredundant intersection of at least two prime ideals is never prime.*

Exercise 6 $\varphi : \mathbf{A}^1 \to \mathbf{A}^3$ *is the polynomial map given by* $t \to (t, t^2, t^3)$. *Show that the image of* φ *is an algebraic variety* $\mathcal{C} \subset \mathbf{A}^3$ *and that* $\varphi : \mathbf{A}^1 \to \mathcal{C}$ *is an isomorphism.*

Exercise 7 *Let* $\mathcal{C} = \mathbf{V}\langle y^2 - x^3\rangle \subset \mathbf{A}^2$. *Show*

1. *that the parametrization* $f : \mathbf{A}^1 \to \mathcal{C}$ *given by* $t \to (t^2, t^3)$ *is a polynomial map;*
2. *that* f *has a rational inverse* $g : \mathcal{C} - \to \mathbf{A}^1$ *given by* $(x,y) \to y/x$;
3. *finally* f *and* g *are inverse isomorphisms between* $\mathbf{A}^1 \setminus \{0\} \cong \mathcal{C} \setminus \{(0,0)\}$.

Exercise 8 *Let* $\mathcal{C} = \mathbf{V}\langle y^3 - x^4 - x^3\rangle \subset \mathbf{A}^2$. *Show that* $(x,y) \to y/x$ *defines a rational map* $\phi : \mathcal{C} - \to \mathbf{A}^1$, *and its inverse is a polynomial map* $f : \mathbf{A}^1 \to \mathcal{C}$ *parametrising* $\mathcal{C}$. *Prove that* f *restricts to an isomorphism* $\mathbf{A}^1 \setminus \{t_1, t_2, t_3\} \cong \mathcal{C} \setminus \{(0,0)\}$.

Exercise 9 *Let* V *be an irreducible variety and let* ϕ *and* φ *be functions in* $k[V]$ *represented by polynomials* f *and* g. *Suppose that* $\phi\dot{\varphi} \equiv 0$ *in* $k[V]$, *but neither* ϕ *nor* φ *is the zero function on* V. *Show that* $V = (V \cap \mathbf{V}\langle f\rangle) \cup (V \cap \mathbf{V}\langle g\rangle)$.

Exercise 10 *Let* $f : \mathbf{A}^1 \to \mathbf{A}^2$ *be given by* $t \to (t, 0)$, *and let* $g : \mathbf{A}^2 - \to \mathbf{A}^1$ *be the rational map given by* $(x,y) \to x/y$. *Show that the composite* $g \cdot f$ *is not defined anywhere. Determine what is the largest subset of the function field* $k(\mathbf{A}^1)$ *on which* $g^* : k[\mathbf{A}^1] \to k(\mathbf{A}^2)$ *is defined.*