Exercises MAI0128 Introduction to Algebraic Geometry. **3** Projective Varieties. Nonsingularity and Dimension

Exercise 1 *Which of the following expressions define rational maps $\varphi : \mathbb{P}^n - \to \mathbb{P}^m$ between projective spaces of the appropriate dimension (n, m= 1 or 2)?. In each case determine domφ, say if φ is birational and in that case describe the inverse map:*

1. $(x, y, z) \to (x, y)$
2. $(x, y, z) \to (1/x, 1/y, 1/z)$
3. $(x, y, z) \to ((x^3 + y^3)/z^3, y^2/z^2, 1)$
4. $(x, y, z) \to (x^2 + y^2, y^2, y^2)$

Exercise 2 *Let $\mathcal{C} \subset \mathbb{P}^3$ be an irreducible curve defined by $\mathcal{C} = \mathcal{Q}_1 \cap \mathcal{Q}_2$, where $\mathcal{Q}_1 = V(wx = q_1), \mathcal{Q}_2 = V(wy = q_2)$, with q_1, q_2 quadratic forms in x, y, z. Show that the projection $\pi : \mathbb{P}^3 - \to \mathbb{P}^2$ defined by $(w, x, y, z, w) \to (x, y, z)$ restrics to an isomorphism of $\mathcal{C}$ with the plane curve given defined by the quadratic form $xq_2 = yq_1$.*

Exercise 3 *Prove that any irreducible quadric $\mathcal{Q} \subset \mathbb{P}^{n+1}$ is rational; i.e. show that if $p \in \mathcal{Q}$ is a non-singular point, then the linear projection of $\mathbb{P}^{n+1}$ to $\mathbb{P}^n$ induces a birational map $\mathbf{Q} - \to \mathbb{P}^n$.*

Exercise 4 *(***Veronese surface***). consider the embedding $\varphi : \mathbb{P}^2 \to \mathbb{P}^5$ given by $(x, y, z) \to (x^2, xy, xz, y^2, yz, z^2)$. Determine the forms defining the image $\mathcal{S} = \varphi(\mathbb{P}^2)$, and show that φ is a isomorphism. Prove that the lines of $\mathbb{P}^2$ are mapped to conics in $\mathbb{P}^5$. For any line $l \subset \mathbb{P}^2$, consider the projective plane $\pi(l) \subset \mathbb{P}^5$ spanned by the conic $\varphi(l)$. Prove that the union of $\pi(l)$; l line in $\mathbf{P}^2$ is a cubic hypersurface in $\mathbb{P}^5$.*

Exercise 5 *The Cartesian product of two affine spaces k^n and k^m is simply another affine space $k^n \times k^m = k^{n+m}$. By considering the corresponding affine space k^n, k^m, k^{n+m} as the $\mathcal{U}_0$ open subset of $\mathbb{P}^n, \mathbb{P}^m, \mathbb{P}^{n+m}$ respectively, show that $\mathbb{P}^{n+m} \neq \mathbb{P}^n \times \mathbb{P}^m$.*

Exercise 6 *The Segre surface of $\mathbb{P}^1 \times \mathbb{P}^1$. Using the embedding $\varphi : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$ given by $((x_0, x_1), (y_0, y_1)) \to (x_0y_0, x_0y_1, x_1y_0, x_1y_1)$, show that $\varphi : \mathbb{P}^1 \times \mathbb{P}^1$ is isomorphic to $\mathcal{S} = V(X_0X_3 - X_1X_2) \subset \mathbb{P}^3$.*

Find equations for the Segre variety of $\mathbb{P}^2 \times \mathbb{P}^1$ in $\mathbb{P}^5$. Which is the intersection of the Segre variety and the Veronese surface in $\mathbb{P}^5$?

Exercise 7 *Find the singular points of the surfaces in k^n (affine space of dim n) given by*

1. $xy^2 = z^2$,

2. $x^2 + y^2 = z^2$;

3. $xy + x^3 + z^3$.

Exercise 8 *Consider the complex numbers $\mathcal{C}$. Show that the projective hypersurface $V_d = V(x_0^d + x_1^d + \cdots + x_n^d) \subset \mathbb{P}^n$ is nonsingular.*

Exercise 9 *Prove that the intersection of an affine hypersurface $V \subset k^n$, not an hyperplane with the tangent hyperplane T_pV is singular at p.*

Exercise 10 *Show how to resolve the following curve singularities by making one or more blow-ups:*

1. $y^3 - x^4$ *and* $y^3 - x^5$

2. $y^2 - x^5$, *in general* $y^2 - x^{2n+1}$

3. $(y^2 - x^2)(y^2 - x^5) - x^8$