More on the Growth Curve model

Martin Singull

Department of Mathematics
Linköping university, Sweden

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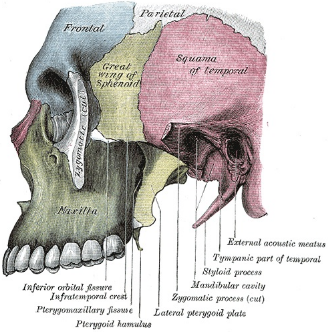
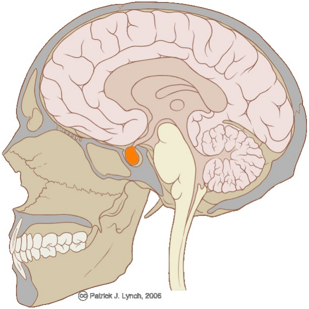
Example - Potthoff & Roy (1964)

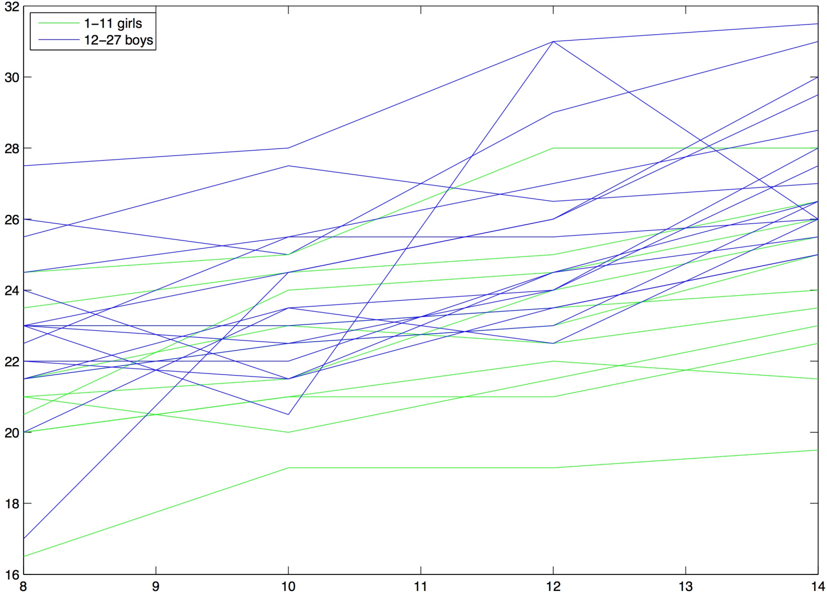
Dental measurements on eleven girls and sixteen boys at four different ages $(8, 10, 12, 14)$ were taken. Each measurement is the distance, in millimeters, from the center of pituitary (hypophysis) to pterygo-maxillary fissure.





$$
\boldsymbol{X} = (\boldsymbol{x}_1, \dots, \boldsymbol{x}_{27})
$$

$$
= \begin{pmatrix}
21 & 21 & 20.5 & 23.5 & 21.5 & 20 & 21.5 & 23 & 20 & \dots \\
16.5 & 24.5 & 26 & 21.5 & 23 & 20 & 25.5 & 24.5 & 22 & \dots \\
\dots & 24 & 23 & 27.5 & 23 & 21.5 & 17 & 22.5 & 23 & 22 \\
20 & 21.5 & 24 & 24.5 & 23 & 21 & 22.5 & 23 & 21 & \dots \\
19 & 25 & 25 & 22.5 & 22.5 & 23.5 & 27.5 & 25.5 & 22 & \dots \\
\dots & 21.5 & 20.5 & 28 & 23 & 23.5 & 24.5 & 25.5 & 24.5 & 21.5 \\
21.5 & 24 & 24.5 & 25 & 22.5 & 21 & 23 & 23.5 & 22 & \dots \\
19 & 28 & 29 & 23 & 24 & 22.5 & 26.5 & 27 & 24.5 & \dots \\
\dots & 24.5 & 31 & 31 & 23.5 & 24 & 26 & 25.5 & 26 & 23.5 \\
23 & 25.5 & 26 & 26.5 & 23.5 & 22.5 & 25 & 24 & 21.5 & \dots \\
19.5 & 28 & 31 & 26.5 & 27.5 & 26 & 27 & 28.5 & 26.5 & \dots \\
\dots & 25.5 & 26 & 31.5 & 25 & 28 & 29.5 & 26 & 30 & 25
\end{pmatrix}.
$$


1–11 girls
12–27 boys
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Growth Curve Model (Potthoff and Roy, 1964)

Definition. Let $\boldsymbol{X} : p \times n$ and $\boldsymbol{B} : q \times k$ be the observation and parameter matrices, respectively, and let $\boldsymbol{A} : p \times q$ and $\boldsymbol{C} : k \times n$ be the within and between individual design matrices, respectively. Suppose that $q \leq p$ and $p \leq n - \mathrm{r}(\boldsymbol{C})$.

The Growth Curve model (GCM) is given by

$$\boldsymbol{X} = \boldsymbol{A}\boldsymbol{B}\boldsymbol{C} + \boldsymbol{E},$$

where $\boldsymbol{E} \sim N_{p,n}(\mathbf{0}, \boldsymbol{\Sigma}, \boldsymbol{I}_n)$.

Example, cont.

Assume that we want to model linear growth, i.e.,

$$\boldsymbol{\mu}_i = \begin{pmatrix} b_{0i} + 8b_{1i} \\ b_{0i} + 10b_{1i} \\ b_{0i} + 12b_{1i} \\ b_{0i} + 14b_{1i} \end{pmatrix}, \quad \text{for } i = 1, 2.$$

For this we may use the parameter and design matrices

$$\boldsymbol{B} = \begin{pmatrix} b_{01} & b_{02} \\ b_{11} & b_{12} \end{pmatrix},$$

$$\boldsymbol{A} = \begin{pmatrix} 1 & 8 \\ 1 & 10 \\ 1 & 12 \\ 1 & 14 \end{pmatrix} \quad \text{and} \quad \boldsymbol{C} = \begin{pmatrix} \mathbf{1}'_{11} & \mathbf{0}'_{16} \\ \mathbf{0}'_{11} & \mathbf{1}'_{16} \end{pmatrix}.$$

Growth Curve Model – MLEs

The MLEs for the parameters $\boldsymbol{B}$ and $\boldsymbol{\Sigma}$ are given by

$$\widehat{\boldsymbol{B}}_{MLE} = \left(\boldsymbol{A}'\boldsymbol{S}^{-1}\boldsymbol{A}\right)^{-}\boldsymbol{A}'\boldsymbol{S}^{-1}\boldsymbol{X}\boldsymbol{C}'\left(\boldsymbol{C}\boldsymbol{C}'\right)^{-} + (\boldsymbol{A}')^{o}\boldsymbol{Z}_1 + \boldsymbol{A}'\boldsymbol{Z}_2\boldsymbol{C}^{o\prime}, \text{ i.e.,}$$

$$\boldsymbol{A}\widehat{\boldsymbol{B}}_{MLE}\boldsymbol{C} = \boldsymbol{P}_{\boldsymbol{A},\boldsymbol{S}}\boldsymbol{X}\boldsymbol{P}_{\boldsymbol{C}'},$$

$$n\widehat{\boldsymbol{\Sigma}}_{MLE} = \left(\boldsymbol{X} - \boldsymbol{A}\widehat{\boldsymbol{B}}_{MLE}\boldsymbol{C}\right)\left(\boldsymbol{X} - \boldsymbol{A}\widehat{\boldsymbol{B}}_{MLE}\boldsymbol{C}\right)' = \underbrace{\widehat{\boldsymbol{R}}_1\widehat{\boldsymbol{R}}_1'}_{=\boldsymbol{S}} + \widehat{\boldsymbol{R}}_2\widehat{\boldsymbol{R}}_2',$$

where

$$\widehat{\boldsymbol{R}}_1 = \boldsymbol{X}\left(\boldsymbol{I}_n - \boldsymbol{P}_{\boldsymbol{C}'}\right),$$

$$\widehat{\boldsymbol{R}}_2 = \left(\boldsymbol{I}_p - \boldsymbol{P}_{\boldsymbol{A},\boldsymbol{S}}\right)\boldsymbol{X}\boldsymbol{P}_{\boldsymbol{C}'},$$

$$\boldsymbol{S} = \widehat{\boldsymbol{R}}_1\widehat{\boldsymbol{R}}_1' = \boldsymbol{X}\left(\boldsymbol{I}_n - \boldsymbol{P}_{\boldsymbol{C}'}\right)\boldsymbol{X}',$$

$$\boldsymbol{P}_{\boldsymbol{C}'} = \boldsymbol{C}'(\boldsymbol{C}\boldsymbol{C}')^{-}\boldsymbol{C} = \text{ projection on } \mathcal{C}(\boldsymbol{C}'),$$

$$\boldsymbol{P}_{\boldsymbol{A},\boldsymbol{S}} = \boldsymbol{A}\left(\boldsymbol{A}'\boldsymbol{S}^{-1}\boldsymbol{A}\right)^{-}\boldsymbol{A}'\boldsymbol{S}^{-1} = \text{ projection on } \mathcal{C}_{\boldsymbol{S}}(\boldsymbol{A}).$$

Growth Curve Model – MLEs

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where

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$$\widehat{\boldsymbol{R}}_2 = \left(\boldsymbol{I}_p - \boldsymbol{P}_{\boldsymbol{A},\boldsymbol{S}}\right)\boldsymbol{X}\boldsymbol{P}_{\boldsymbol{C}'},$$

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$$
\begin{aligned}
&\mathcal{C}_{\boldsymbol{S}}(\boldsymbol{A}) \otimes \mathcal{C}(\boldsymbol{C}') \boxplus (\mathcal{C}_{\boldsymbol{S}}(\boldsymbol{A}) \otimes \mathcal{C}(\boldsymbol{C}'))^{\perp} \\
&\quad = (\mathcal{C}_{\boldsymbol{S}}(\boldsymbol{A}) \otimes \mathcal{C}(\boldsymbol{C}')) \boxplus \mathcal{C}_{\boldsymbol{S}}(\boldsymbol{A})^{\perp} \otimes \mathcal{C}(\boldsymbol{C}') \boxplus \mathcal{V} \otimes \mathcal{C}(\boldsymbol{C}')^{\perp}
\end{aligned}
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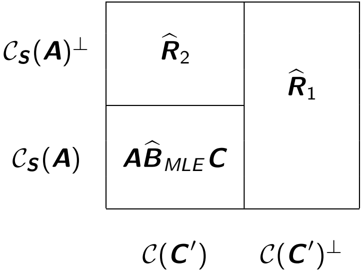


$$\widehat{\boldsymbol{R}}_1 = \boldsymbol{X}(\boldsymbol{I}_n - \boldsymbol{P}_{\boldsymbol{C}'})$$

$$\widehat{\boldsymbol{R}}_2 = (\boldsymbol{I}_p - \boldsymbol{P}_{\boldsymbol{A},\boldsymbol{S}})\boldsymbol{X}\boldsymbol{P}_{\boldsymbol{C}'}$$

$$
\begin{aligned}
\boldsymbol{A}\widehat{\boldsymbol{B}}_{MLE}\boldsymbol{C} &= \boldsymbol{P}_{\boldsymbol{A},\boldsymbol{S}}\boldsymbol{X}\boldsymbol{P}_{\boldsymbol{C}'}, \\
n\widehat{\boldsymbol{\Sigma}}_{MLE} &= \widehat{\boldsymbol{R}}_1\widehat{\boldsymbol{R}}_1' + \widehat{\boldsymbol{R}}_2\widehat{\boldsymbol{R}}_2'.
\end{aligned}
$$

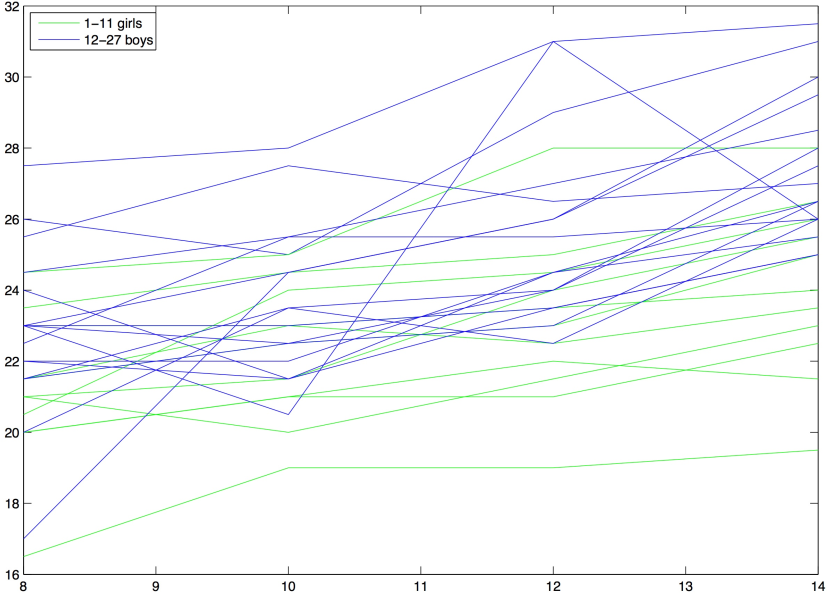
Example - Potthoff & Roy (1964), cont.

The MLEs for the Example are given by

$$\widehat{\boldsymbol{B}}_{MLE} = \begin{pmatrix} 17.4254 & 15.8423 \\ 0.4764 & 0.8268 \end{pmatrix}$$

and

$$\widehat{\boldsymbol{\Sigma}} = \begin{pmatrix} 5.1192 & 2.4409 & 3.6105 & 2.5222 \\ 2.4409 & 3.9279 & 2.7175 & 3.0623 \\ 3.6105 & 2.7175 & 5.9798 & 3.8235 \\ 2.5222 & 3.0623 & 3.8235 & 4.6180 \end{pmatrix}.$$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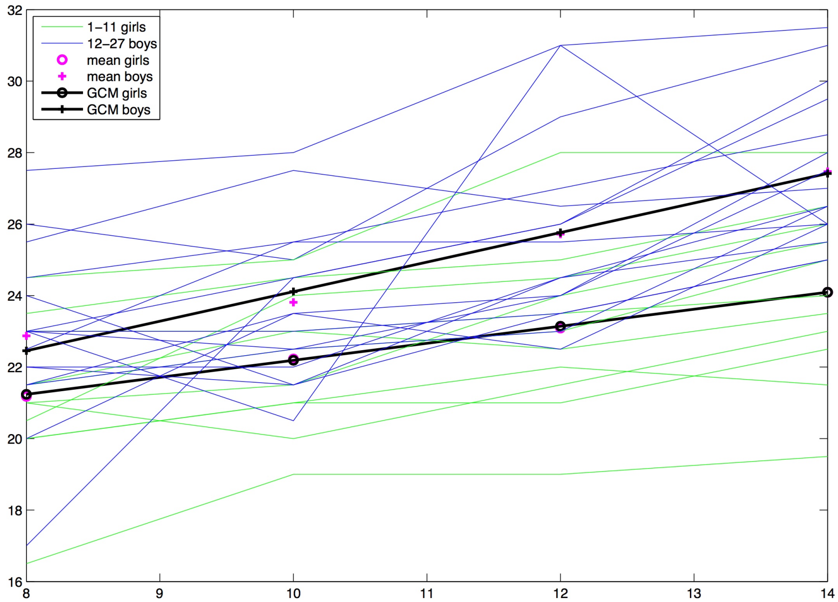
1–11 girls
12–27 boys
mean girls
mean boys
GCM girls
GCM boys
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Properties of the Estimators

Let $\mathrm{r}(A) = q$ and $\mathrm{r}(C) = k$, the mean and covariance for $\widehat{\boldsymbol{B}}_{MLE}$ are (Kollo and von Rosen, 2005)

$$\mathrm{E}\left(\widehat{\boldsymbol{B}}_{MLE}\right) = \boldsymbol{B}, \qquad \text{and}$$

$$\mathrm{D}\left(\widehat{\boldsymbol{B}}_{MLE}\right) = \frac{n-k-1}{n-k-1-(p-q)}(\boldsymbol{C}\boldsymbol{C}')^{-1} \otimes (\boldsymbol{A}'\boldsymbol{\Sigma}^{-1}\boldsymbol{A})^{-1},$$

if $n-k-1-(p-q) > 0$, and

$$\mathrm{E}\left(\widehat{\boldsymbol{\Sigma}}_{MLE}\right) = \boldsymbol{\Sigma} - \frac{k}{n}\frac{n-k-1-2(p-q)}{n-k-1-(p-q)}\boldsymbol{A}(\boldsymbol{A}'\boldsymbol{\Sigma}^{-1}\boldsymbol{A})^{-1}\boldsymbol{A}'.$$

The bias depends on the design $\boldsymbol{A}$ and thus it could be significant.

Also, note that since $q \leq p \leq n$ we have $\dfrac{n-k-1}{n-k-1-(p-q)} \geq 1.$

Example - p Time Points and n Observations

In a small simulation example we may use the parameters and designs

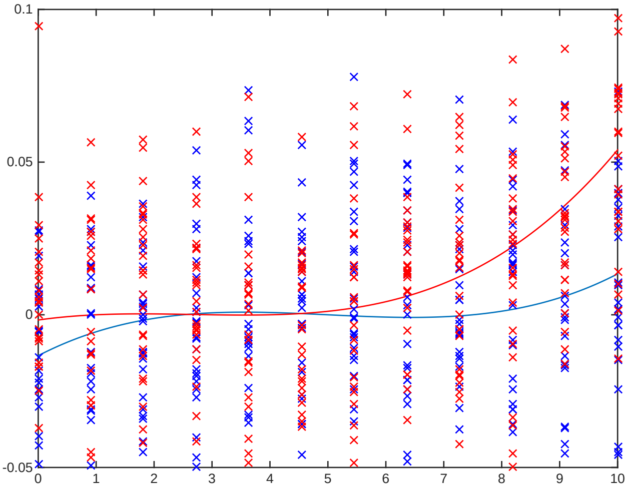
$$\boldsymbol{B} = \begin{pmatrix} b_{01} & b_{02} \\ b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix} = \begin{pmatrix} -0.0134 & -0.0017 \\ 0.0098 & 0.0027 \\ -0.0021 & -0.0011 \\ 0.0001 & 0.0001 \end{pmatrix},$$

$$\boldsymbol{A} = \begin{pmatrix} 1 & t_1 & t_1^2 & t_1^3 \\ 1 & t_2 & t_2^2 & t_2^3 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & t_p & t_p^2 & t_p^3 \end{pmatrix} \quad \text{and} \quad \boldsymbol{C} = \begin{pmatrix} \mathbf{1}'_{n_1} & \mathbf{0}'_{n_2} \\ \mathbf{0}'_{n_1} & \mathbf{1}'_{n_2} \end{pmatrix},$$

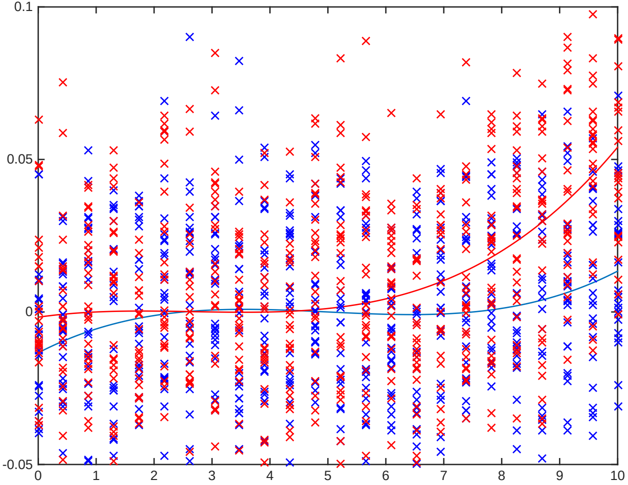
where we have used $q = 4$ (i.e., cubic growth) and $k = 2$ groups for simplicity with $n_1 = n_2 = n/2$. Furthermore, we put $t_1 = 0$ and

$$t_i = i\frac{10}{p-1}, \quad \text{for } i = 1, ..., p-1.$$

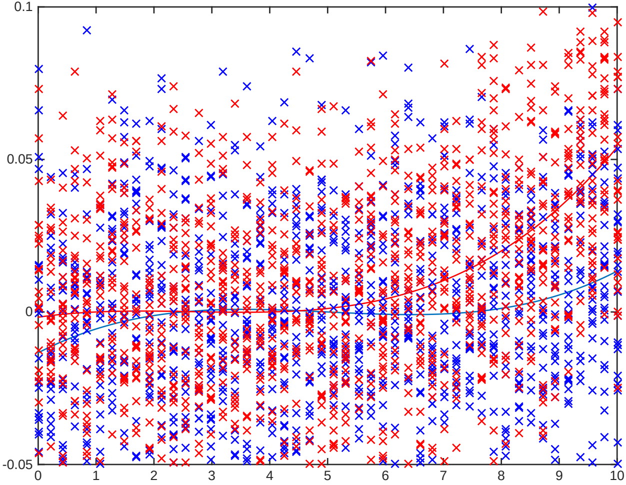
$n_1 = n_2 = 25$ with $p = 12 \leq n - k = 48$



$n_1 = n_2 = 25$ with $p = 24 \leq n - k = 48$



$n_1 = n_2 = 25$ with $p = 48 \leq n - k = 48$



Unweighted Estimator of $\boldsymbol{B}$

A natural alternative, proposed by Srivastava & Singull (2017a,b), to the MLE would be an unweighted estimator of $\boldsymbol{B}$ given by

$$\widehat{\boldsymbol{B}} = (\boldsymbol{A}'\boldsymbol{A})^{-1}\boldsymbol{A}'\boldsymbol{X}\boldsymbol{C}'(\boldsymbol{C}\boldsymbol{C}')^{-1}.$$

This estimator is simpler than the MLE, since we do not need to calculate the inverse of the sum of squares matrix $\boldsymbol{S}^{-1}$.

The distribution of the estimator is given by

$$\widehat{\boldsymbol{B}} \sim N_{q,m}(\boldsymbol{B}, (\boldsymbol{A}'\boldsymbol{A})^{-1}\boldsymbol{A}'\boldsymbol{\Sigma}\boldsymbol{A}(\boldsymbol{A}'\boldsymbol{A})^{-1}, (\boldsymbol{C}\boldsymbol{C}')^{-1}),$$

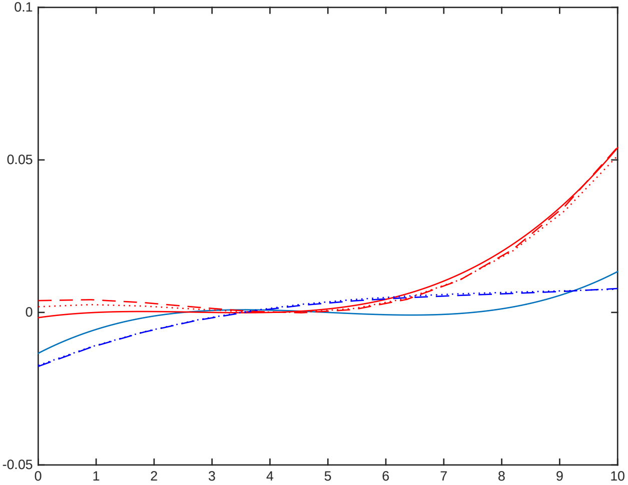
i.e., we have

$$\mathrm{E}\left(\widehat{\boldsymbol{B}}\right) = \boldsymbol{B}, \quad \text{and} \quad \boxed{\mathrm{D}\left(\widehat{\boldsymbol{B}}\right) = (\boldsymbol{C}\boldsymbol{C}')^{-1} \otimes (\boldsymbol{A}'\boldsymbol{A})^{-1}\boldsymbol{A}'\boldsymbol{\Sigma}\boldsymbol{A}(\boldsymbol{A}'\boldsymbol{A})^{-1}.}$$

Example, cont.

$N_1 = N_2 = 25$ with $p = 12 \leq n - k = 48$

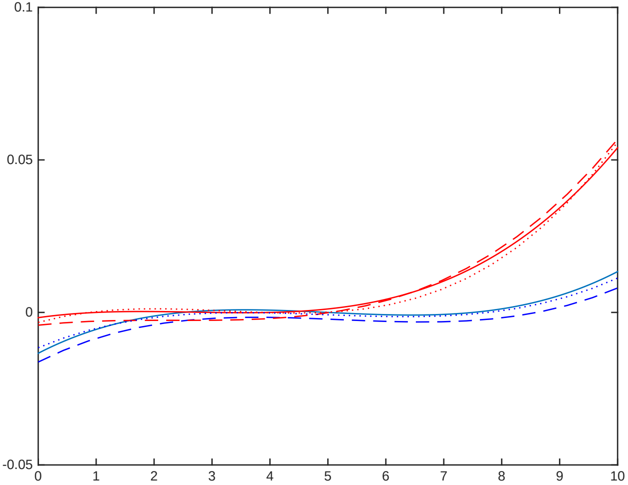
(—— real growth, $- - -$ weighted est. (MLE) and $\cdots$ unweighted est.)



Example, cont.

$N_1 = N_2 = 25$ with $p = 24 \leq n - k = 48$

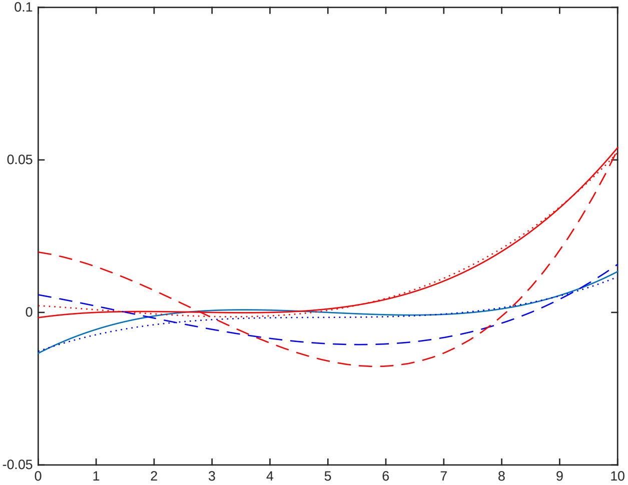
(—— real growth, – – – weighted est. (MLE) and $\cdots$ unweighted est.)



Example, cont.

$N_1 = N_2 = 25$ with $p = 48 \leq n - k = 48$

(—— real growth, – – – weighted est. (MLE) and $\cdots$ unweighted est.)



Example - Potthoff & Roy (1964), cont.

The two different estimates are

$$\widehat{\boldsymbol{B}}_{MLE} = \begin{pmatrix} 17.4254 & 15.8423 \\ 0.4764 & 0.8268 \end{pmatrix}$$

and

$$\widehat{\boldsymbol{B}} = \begin{pmatrix} 17.3727 & 16.3406 \\ 0.4795 & 0.7844 \end{pmatrix}.$$

Compare the Estimators $\widehat{\boldsymbol{B}}_{MLE}$ and $\widehat{\boldsymbol{B}}$

Both $\widehat{\boldsymbol{B}}_{MLE}$ and $\widehat{\boldsymbol{B}}$ are unbiased and

$$\mathrm{D}\left(\widehat{\boldsymbol{B}}_{MLE}\right) = \frac{n-k-1}{n-k-1-(p-q)}(\boldsymbol{C}\boldsymbol{C}')^{-1} \otimes (\boldsymbol{A}'\boldsymbol{\Sigma}^{-1}\boldsymbol{A})^{-1},$$

$$\mathrm{D}\left(\widehat{\boldsymbol{B}}\right) = (\boldsymbol{C}\boldsymbol{C}')^{-1} \otimes (\boldsymbol{A}'\boldsymbol{A})^{-1}\boldsymbol{A}'\boldsymbol{\Sigma}\boldsymbol{A}(\boldsymbol{A}'\boldsymbol{A})^{-1}.$$

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$$\mathrm{D}\left(\widehat{\boldsymbol{B}}\right) = (\boldsymbol{C}\boldsymbol{C}')^{-1} \otimes (\boldsymbol{A}'\boldsymbol{A})^{-1}\boldsymbol{A}'\boldsymbol{\Sigma}\boldsymbol{A}(\boldsymbol{A}'\boldsymbol{A})^{-1}.$$

Hence, we need to study

$$(\boldsymbol{A}'\boldsymbol{\Sigma}^{-1}\boldsymbol{A})^{-1} \quad \text{versus} \quad (\boldsymbol{A}'\boldsymbol{A})^{-1}\boldsymbol{A}'\boldsymbol{\Sigma}\boldsymbol{A}(\boldsymbol{A}'\boldsymbol{A})^{-1}.$$

Following Rao (1967) (Lemma 2.c) or Baksalary and Puntanen (1991) one can show that

$$(\boldsymbol{A}'\boldsymbol{\Sigma}^{-1}\boldsymbol{A})^{-1} \leq (\boldsymbol{A}'\boldsymbol{A})^{-1}\boldsymbol{A}'\boldsymbol{\Sigma}\boldsymbol{A}(\boldsymbol{A}'\boldsymbol{A})^{-1}$$

with equality if and only if $\mathcal{C}(\boldsymbol{\Sigma}^{-1}\boldsymbol{A}) = \mathcal{C}(\boldsymbol{A})$ (MLE = unweighted).

The inequality is with respect to the Loewner partial ordering, i.e., $\boldsymbol{A} \leq \boldsymbol{B}$ if $\boldsymbol{B} - \boldsymbol{A}$ is nonnegative definite.

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For large n, the unweighted unbiased estimator of $\boldsymbol{B}$ *has a larger covariance than the weighted one, as expected since the weighted estimator is the MLE.*

But, when also p is large, but still less than n, the factor

$$\boxed{1 \ll (n-1)/(n-1-(p-q)) \quad \Rightarrow \quad \mathrm{D}\left(\widehat{\boldsymbol{B}}\right) < \mathrm{D}\left(\widehat{\boldsymbol{B}}_{MLE}\right)}$$

$\mathcal{C}(\boldsymbol{\Sigma}^{-1}\boldsymbol{A}) = \mathcal{C}(\boldsymbol{A})$

Under the restriction $\mathcal{C}(\boldsymbol{\Sigma}^{-1}\boldsymbol{A}) = \mathcal{C}(\boldsymbol{A})$, the MLE for the GCM is given by the unweighted estimator.

This condition is fulfilled, for example when

- sphericity $\boldsymbol{\Sigma} = \sigma^2 \boldsymbol{I}_p$, or
- intraclass covariance matrix $\boldsymbol{\Sigma} = \sigma^2((1-\rho)\boldsymbol{I}_p + \rho\mathbf{1}\mathbf{1}')$ and $\boldsymbol{A}$ includes a column vector of ones, e.g., $\boldsymbol{A} = [\mathbf{1} \ : \ \boldsymbol{A}_1]$.

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