Lösningsskiss till tentamen i Matematisk analys, del 2, 764G07, TEN-2, 2020-01-17.

**1** **a)** $\int x^2 \ln x \, dx \;\Big/ \begin{matrix} PI \\ g = \ln x \Rightarrow g' = \frac{1}{x} \\ f = x^2 \Rightarrow F = \frac{x^3}{3} \end{matrix} \Big/ = \ln x \cdot \frac{x^3}{3} - \int \frac{1}{x} \cdot \frac{x^3}{3} dx =$

$= \frac{x^3}{3} \ln x - \frac{x^3}{9} + C$

**b)** $\int \frac{\sin \sqrt{x}}{\sqrt{x}} dx \;\Big/ \begin{matrix} t = \sqrt{x} \\ x = t^2, \; dx = 2t\,dt \end{matrix} \Big/ = \int \frac{\sin t}{t} \cdot 2t\,dt =$

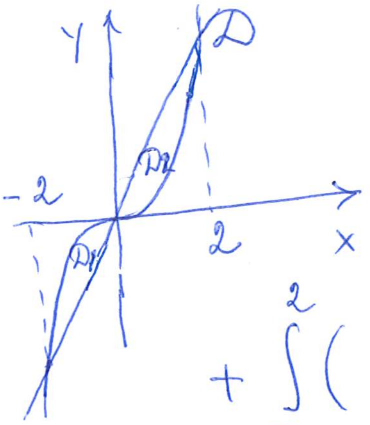
$= 2 \int \sin t \, dt = -2\cos t + C = -2\cos\sqrt{x} + C$

**c)** $\int \frac{x-5}{x^2-5x+6} dx = \int \frac{x-5}{(x-2)(x-3)} dx \;\Big/ \frac{x-5}{(x-2)(x-3)} = \frac{3}{x-2} - \frac{2}{x-3} \Big/$

$= \int \left( \frac{3}{x-2} - \frac{2}{x-3} \right) dx = 3 \ln|x-2| - 2 \ln|x-3| + C.$

Svar a) $\frac{x^3}{3} \ln x - \frac{x^3}{9} + C$ b) $-2\cos\sqrt{x} + C$. c) $\ln \frac{|x-2|^3}{(x-3)^2} + C$

**2** $I = \iint_D xy \, dx\,dy \;\Big/ \begin{matrix} 4x = x^3 \\ \Updownarrow \\ x^3 - 4x = 0 \\ \Updownarrow \\ x(x^2-4) = 0 \\ x = 0, \; x = \pm 2 \end{matrix} \Big/ = \iint_{D_1} xy \, dx\,dy +$




$+ \iint_{D_2} xy \, dx\,dy = \int_{-2}^{0} \left( \int_{4x}^{x^3} xy \, dy \right) dx +$

$+ \int_0^2 \left( \int_{x^3}^{4x} xy \, dy \right) dx = \int_{-2}^{0} x \left( \frac{y^2}{2} \right)_{4x}^{x^3} dx + \int_0^2 x \left( \frac{y^2}{2} \right)_{x^3}^{4x} dx =$

$= \frac{1}{2} \int_{-2}^{0} (x^7 - 16x^3) dx + \frac{1}{2} \int_0^2 (16x^3 - x^7) dx = \frac{1}{2} \left( \frac{x^8}{8} - 4x^4 \right)_{-2}^{0} +$

$+ \frac{1}{2} \left( 4x^4 - \frac{x^8}{8} \right)_0^2 = -\frac{1}{2} \cdot \frac{2^8}{8} + 2 \cdot 2^4 + 2 \cdot 2^4 - \frac{1}{2} \cdot \frac{2^8}{8} = 32$

Svar: $I = 32$

3 a $f(x)=\ln(1+\sin x)=\ln\left(1+x-\frac{x^3}{3!}+O(x^5)\right)=$

$=\Big/ \ln(1+t)=t-\frac{t^2}{2}+\frac{t^3}{3}+O(t^4)\Big/= x-\frac{x^3}{3!}+O(x^5)-$

$-\frac{1}{2}\left(x-\frac{x^3}{3!}+O(x^5)\right)^2+\frac{1}{3}\left(x-\frac{x^3}{3!}+O(x^5)\right)^3+O(x^4)=$

$=x-\frac{x^3}{3!}+O(x^5)-\frac{1}{2}\left(x^2+O(x^4)\right)+\frac{1}{3}\left(x^3+O(x^5)\right)+O(x^4)=$

$=x-\frac{1}{2}x^2+\frac{1}{6}x^3+O(x^4)$ Svar: $p_3(x)=x-\frac{x^2}{2}+\frac{x^3}{6}$.

b $\lim\limits_{x\to 0}\frac{\ln(\cos x)}{e^x-1-x}=\lim\limits_{x\to 0}\frac{\ln\left(1-\frac{x^2}{2}+O(x^4)\right)}{1+x+\frac{x^2}{2}+O(x^3)-1-x}$

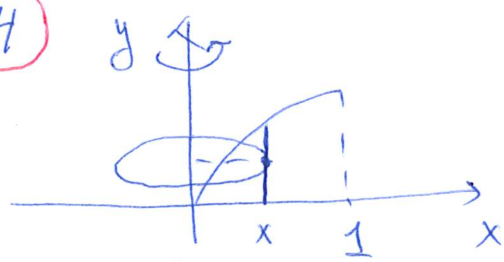
$=\lim\limits_{x\to 0}\frac{-\frac{x^2}{2}+O(x^4)+O(x^4)}{\frac{x^2}{2}+O(x^3)}=\lim\limits_{x\to 0}\frac{x^2\left(-\frac{1}{2}+O(x^2)\right)}{x^2\left(\frac{1}{2}+O(x)\right)}=-1$

c $f(x)=x\sin(2x+x^2)-2x^2-x^3=x\left(2x+x^2-\frac{(2x+x^2)^3}{3!}+\right.$
$\left.+O\left((2x+x^2)^5\right)\right)-2x^2-x^3=x\left(2x+x^2-\frac{1}{3!}\left(8x^3+O(x^4)\right)+\right.$
$\left.+O(x^5)\right)-2x^2-x^3=-\frac{8}{3!}x^4+O(x^5)=-\frac{4}{3}x^4+O(x^5)$

$\Rightarrow f(x)-f(0)=-\frac{4}{3}x^4+O(x^5)<0$ för alla $x$ nära 0 dvs $f(x)<f(0)$ för alla $x$ nära 0

Svar $x=0$ - maximipunkt.

4




$dV=2\pi x\cdot f(x)\,dx=2\pi x\frac{\sqrt{x}}{x+1}\,dx$

$V=2\pi\int_0^1\frac{x\sqrt{x}}{x+1}\,dx\ \Bigg/\begin{array}{l}x=t^2\\ dx=2t\,dt\\ x=0\Rightarrow\alpha=0\\ x=1\Rightarrow\beta=1\end{array}\Bigg/=$

$=2\pi\int_0^1\frac{t^2\cdot t}{t^2+1}\cdot 2t\,dt=2\pi\int_0^1\frac{2t^4}{t^2+1}\,dt=4\pi\int_0^1\left(t^2-1+\frac{1}{t^2+1}\right)dt=$

$=4\pi\left(\frac{1}{3}t^3-t+\arctan t\right)\Big|_0^1=4\pi\left(\frac{1}{3}-1+\frac{\pi}{4}\right)=\frac{\pi}{3}(3\pi-8)$ (v.e.)

Svar $V=\pi^2-\frac{8}{3}\pi$ v.e.

5 a $y = \frac{x}{\ln x} \Rightarrow y' = \frac{\ln x - x \cdot \frac{1}{x}}{\ln^2 x} = \frac{\ln x - 1}{\ln^2 x}$

$VL = x^2 \cdot \frac{\ln x - 1}{\ln^2 x}$, $HL = \frac{x^2}{\ln x} - \frac{x^2}{\ln^2 x} = x^2 \frac{\ln x - 1}{\ln^2 x}$

Vi ser att $VL = HL$ för alla $x > 0 \Rightarrow$

$y = \frac{x}{\ln x}$ är en lösning till $x^2 y' = xy - y^2$.

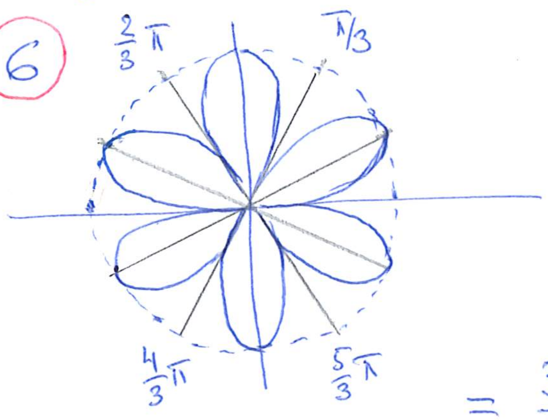
6 $y' + xy = e^{-\frac{x^2}{2}}$. $IF = e^{\frac{x^2}{2}} \Rightarrow$

$y' e^{\frac{x^2}{2}} + x e^{\frac{x^2}{2}} y = 1 \Leftrightarrow \left(y e^{\frac{x^2}{2}}\right)' = 1 \Leftrightarrow$

$y e^{\frac{x^2}{2}} = x + C \Leftrightarrow y = x e^{-\frac{x^2}{2}} + C e^{-\frac{x^2}{2}}$.

$y(0) = 1 \Rightarrow 1 = C$ dvs $y = x e^{-\frac{x^2}{2}} + e^{-\frac{x^2}{2}}$ - svar.

6




$A = \frac{1}{2}\int_0^{2\pi} r^2 \, d\theta = 6 \cdot \frac{1}{2} \int_0^{\pi/3} \sin^4 3\theta \, d\theta =$

$= 3 \int_0^{\pi/3} \left(\frac{1 - \cos 6\theta}{2}\right)^2 d\theta =$

$= \frac{3}{4} \int_0^{\pi/3} \left(1 - 2\cos 6\theta + \frac{1 + \cos 12\theta}{2}\right) d\theta =$

$= \frac{3}{4}\left(\theta - \frac{2}{6}\sin 6\theta + \frac{1}{2}\theta + \frac{1}{2}\frac{\sin 12\theta}{12}\right)\Big|_0^{\pi/3} = \frac{3}{4} \cdot \frac{3}{2} \cdot \frac{\pi}{3} = \frac{3}{8}\pi$ (a.e.)

Svar: $A = \frac{3}{8}\pi$ a.e.

7 $\lim_{x \to \infty} \frac{\ln(3x+2) + \ln(3x-2) - 2\ln x + A}{\arctan\left(1 - \cos\frac{1}{x}\right)}$ $\left/ \begin{array}{c} t = \frac{1}{x} \\ t \to 0 \text{ då } x \to \infty \end{array} \right/ =$

$= \lim_{t \to 0} \frac{\ln(9 - 4t^2) + A}{\arctan(1 - \cos t)} = \lim_{t \to 0} \frac{\ln 9 + A + \ln\left(1 - \frac{4}{9}t^2\right)}{\arctan\left(1 - \left(1 - \frac{t^2}{2} + O(t^4)\right)\right)} =$

$= \lim_{t \to 0} \frac{\ln 9 + A - \frac{4}{9}t^2 + O(t^4)}{\frac{t^2}{2} + O(t^6)} < \infty$ om $A = -\ln 9$ då har vi

$\lim_{t \to 0} \frac{-\frac{4}{9}t^2 + O(t^4)}{\frac{1}{2}t^2 + O(t^6)} = \frac{-\frac{4}{9}}{1/2} = -\frac{8}{9}$

Svar: $A = -\ln 9$, $-\frac{8}{9}$.