Lösningsskiss till tentamen i Matematisk analys 764G07, del2

2021-01-12

1a $\displaystyle\int (x+1)\cos(3x)\,dx \Big/ \begin{array}{l} \text{PI} \\ g = x+1 \Rightarrow g' = 1 \\ f' = \cos 3x \Rightarrow f = \frac{1}{3}\sin 3x \end{array} \Big/ = \frac{1}{3}(x+1)\sin 3x - \frac{1}{3}\int \sin 3x\,dx =$

$= \frac{1}{3}(x+1)\sin 3x + \frac{1}{9}\cos 3x + C$

1b $\displaystyle\int \frac{\sqrt{2+x}}{x+3}\,dx \Big/ \begin{array}{l} t = \sqrt{2+x} \Rightarrow t^2 = 2+x \\ 2t\,dt = dx \end{array} \Big/ = \int \frac{t}{t^2+1}\,2t\,dt = 2\int \frac{t^2}{t^2+1}\,dt$

$= 2\displaystyle\int \left(1 - \frac{1}{t^2+1}\right) dt = 2\left(t - \arctan t\right) + C =$

$= 2\sqrt{2+x} - 2\arctan\sqrt{2+x} + C$

1c $\displaystyle\int \frac{\cos x}{\sin^2 x + 2\sin x + 5}\,dx = \Big/ \begin{array}{l} t = \sin x \\ dt = \cos x\,dx \end{array} \Big/ = \int \frac{dt}{t^2+2t+5} =$

$= \displaystyle\int \frac{dt}{(t+1)^2+4} = \frac{1}{4}\int \frac{dt}{\left(\frac{t+1}{2}\right)^2+1} = \Big/ \begin{array}{l} s = \frac{t+1}{2} \\ ds = \frac{1}{2}dt \end{array} \Big/ =$

$= \frac{1}{4}\cdot 2\displaystyle\int \frac{ds}{s^2+1} = \frac{1}{2}\arctan s + C = \frac{1}{2}\arctan\frac{\sin x + 1}{2} + C$

Svar: a) $\frac{1}{3}(x+1)\sin 3x + \frac{1}{3}\cos 3x + C$

b) $2\sqrt{2+x} - 2\arctan\sqrt{2+x} + C$

c) $\frac{1}{2}\arctan\frac{\sin x + 1}{2} + C$

2 $(x-1)e^{2x}y' + \frac{e^{2x}}{x}y = x^2, \quad x > 1 \iff y' + \frac{1}{x(x-1)}y = \frac{x^2}{(x-1)e^{2x}}, \quad x > 1$

- $f(x) = \frac{1}{x(x-1)} \Rightarrow F(x) = \displaystyle\int \frac{1}{x(x-1)}\,dx = \int\left(\frac{1}{x-1} - \frac{1}{x}\right)dx =$

$= \ln|x-1| - \ln|x| + C = \ln\left|\frac{x-1}{x}\right| + C \Rightarrow$

$IF = \frac{x-1}{x}$

- $\frac{x-1}{x}y' + \frac{1}{x^2}y = xe^{-2x} \Rightarrow \left(\frac{x-1}{x}y\right)' = xe^{-2x} \Rightarrow \frac{x-1}{x}y = \displaystyle\int xe^{-2x}\,dx =$

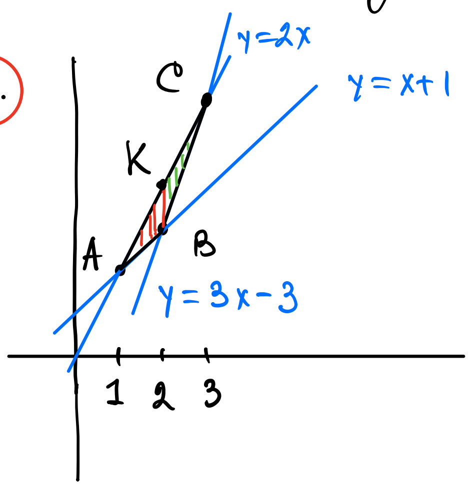
$$\left/ \begin{matrix} g \overset{PI}{=} x \Rightarrow g' = 1 \\ f' = e^{-2x} \Rightarrow f = -\frac{1}{2} e^{-2x} \end{matrix} \right/ = -\frac{x}{2} e^{-2x} + \int \frac{e^{-2x}}{2}\, dx = -\frac{x}{2} e^{-2x} - \frac{1}{4} e^{-2x} + C$$

- $y = -\dfrac{x^2}{2(x-1)} e^{-2x} - \dfrac{x}{4(x-1)} e^{-2x} + C\dfrac{x}{x-1}$ – samtliga lösningar

- $\lim\limits_{x\to\infty} y(x) = 1 \Rightarrow \lim\limits_{x\to\infty} \Big(\underbrace{-\dfrac{x^2}{2(x-1)} e^{-2x}}_{\to 0} - \underbrace{\dfrac{x}{4(x-1)} e^{-2x}}_{\to 0} + \dfrac{Cx}{x-1} \Big) = C = 1$
enligt hastighetstabell.

Svar: $y = -\dfrac{x^2}{2(x-1)} e^{-2x} - \dfrac{x}{4(x-1)} e^{-2x} + \dfrac{x}{x-1}$

3.



$A: \begin{cases} y = x+1 \\ y = 2x \end{cases} \Rightarrow \begin{cases} x = 1 \\ y = 2 \end{cases} \Rightarrow A = (1,2)$

$B: \begin{cases} y = x+1 \\ y = 3x-3 \end{cases} \Rightarrow \begin{cases} 3x-3 = x+1 \Rightarrow x = 2 \\ y = 3 \end{cases} \Rightarrow B = (2,3)$

$C: \begin{cases} y = 2x \\ y = 3x-3 \end{cases} \Rightarrow \begin{cases} x = 3 \\ y = 6 \end{cases} \Rightarrow C = (3,6)$

$$A = \iint\limits_{\Delta ABC} dx\,dy = \iint\limits_{\Delta ABK} dx\,dy + \iint\limits_{\Delta BCK} dx\,dy = \int_1^2 \Big(\int_{x+1}^{2x} dy \Big) dx +$$

$$+ \int_2^3 \Big(\int_{3x-3}^{2x} dy \Big) dx = \int_1^2 (2x - (x+1))\, dx + \int_2^3 (2x - (3x-3))\, dx =$$

$$= \int_1^2 (x-1)\, dx + \int_2^3 (-x+3)\, dx = \Big(\frac{x^2}{2} - x \Big)_1^2 + \Big(-\frac{x^2}{2} + 3x \Big)_2^3 =$$

$$= \Big(2 - 2 - \big(\tfrac{1}{2} - 1\big)\Big) + \Big(-\frac{9}{2} + 2 + (9-6) \Big) = \frac{1}{2} - \frac{9}{2} + 5 = 1 \text{ (a.e.)}$$

Svar: $A = 1$ a.e.

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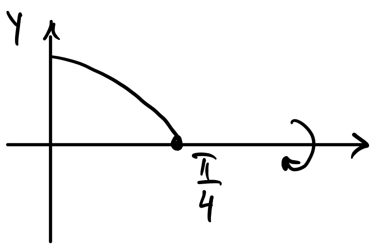


$y = \cos(2x), \quad 0 \le x \le \frac{\pi}{4}.$

$dV = \pi f^2(x)\, dx = \pi \cos^2(2x)\, dx$

$$V=\int_0^{\pi/4} dV=\pi\int_0^{\pi/4}\cos^2(2x)\,dx=\pi\int_0^{\pi/4}\frac{1+\cos(4x)}{2}\,dx=$$

$$=\frac{\pi}{2}\left(x+\frac{\sin(4x)}{4}\right)\Big|_0^{\pi/4}=\frac{\pi}{2}\cdot\frac{\pi}{4}=\frac{\pi^2}{8}\ (\text{v.e.})$$

<u>Svar:</u> $V=\frac{\pi^2}{8}$ v.e.

5a

$$\lim_{x\to 0}\frac{x+\cos(2x)-e^x}{\sin(3x^2)}=\lim_{x\to 0}\frac{x+\cos(2x)-e^x}{3x^2+O(x^6)}=$$

$$=\lim_{x\to 0}\frac{x+\left(1-\frac{(2x)^2}{2}+O(x^4)\right)-\left(1+x+\frac{x^2}{2}+O(x^3)\right)}{3x^2+O(x^6)}=$$

$$=\lim_{x\to 0}\frac{-2x^2+O(x^4)-\frac{1}{2}x^2-O(x^3)}{3x^2+O(x^6)}=\lim_{x\to 0}\frac{-\frac{5}{2}x^2+O(x^3)}{3x^2+O(x^6)}=$$

$$=\lim_{x\to 0}\frac{x^2\left(-\frac{5}{2}+O(x)\right)}{x^2\left(3+O(x^4)\right)}=\frac{-5/2}{3}=-\frac{5}{6}.$$

5b

$$\lim_{x\to 2}\frac{\ln(2x-3)}{\sin(x-2)}\ \Big/\ \begin{matrix}t=x-2\\ t\to 0 \text{ då } x\to 2\end{matrix}\ \Big/=\lim_{t\to 0}\frac{\ln(2t+1)}{\sin t}=$$

$$=\lim_{t\to 0}\frac{2t+O(t^2)}{t+O(t^3)}=\lim_{t\to 0}\frac{t\,(2+O(t))}{t\,(1+O(t^2))}=2$$

5c

$$\int_2^{\infty}\frac{dx}{x^2-1}=\lim_{b\to\infty}\int_2^{b}\frac{dx}{(x-1)(x+1)}=\Big/\frac{1}{(x-1)(x+1)}=\frac{\frac{1}{2}}{x-1}-\frac{\frac{1}{2}}{x+1}\Big/$$

$$=\lim_{b\to\infty}\left[\frac{1}{2}\ln|x-1|-\frac{1}{2}\ln|x+1|\right]_2^b=\lim_{b\to\infty}\frac{1}{2}\left(\ln\left|\frac{x-1}{x+1}\right|\right)_2^b=$$

$$=\lim_{b\to\infty}\frac{1}{2}\left(\ln\frac{b-1}{b+1}-\ln\frac{1}{3}\right)\ \Big/\ \lim_{b\to\infty}\frac{b-1}{b+1}=\lim_{b\to\infty}\frac{b\left(1-\frac{1}{b}\right)}{b\left(1+\frac{1}{b}\right)}=$$

$$= \lim_{b\to\infty} \frac{1-\frac{1}{b}}{1+\frac{1}{b}} = 1 \Big/ = \frac{1}{2}\Big(\underbrace{\ln 1}_{=0} - \ln\frac{1}{3}\Big) = \frac{1}{2}\ln 3$$

Svar: a) $-\frac{5}{6}$ b) 2 c) $\frac{1}{2}\ln 3$

6 $$f(x) = \ln(\cos x) + \frac{x^2}{2} = \ln\Big(1-\frac{x^2}{2}+\frac{x^4}{4!}+O(x^6)\Big)+\frac{x^2}{2} =$$

$$= \Big/ \ln(1+t) = t - \frac{t^2}{2} + O(t^3) \text{ med } t = -\frac{x^2}{2}+\frac{x^4}{4!}+O(x^6),$$

$$t^2 = \Big(-\frac{x^2}{2}+\frac{x^4}{4!}+O(x^6)\Big)\Big(-\frac{x^2}{2}+\frac{x^4}{4!}+O(x^6)\Big) =$$

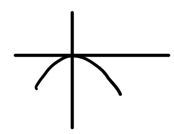
$$= \frac{x^4}{4} + O(x^6), \quad O(t^3) = O(x^6) \Big/ \Rightarrow$$

$$f(x) = -\frac{x^2}{2} + \frac{x^4}{4!} + O(x^6) - \frac{1}{2}\Big(\frac{x^4}{4}+O(x^6)\Big) + O(x^6) + \frac{x^2}{2} =$$

$$= -\frac{1}{12}x^4 + O(x^6)$$

$$f(x) = f(x) - f(0) = -\frac{1}{12}x^4 + O(x^6) < 0 \text{ för alla } x \text{ nära } 0$$

$\Rightarrow$ $f(x) < f(0)$ för alla x nära 0



$\Rightarrow$ $x=0$ är lokal maximipunkt

Svar: f har lokalt maximum i $x=0$.

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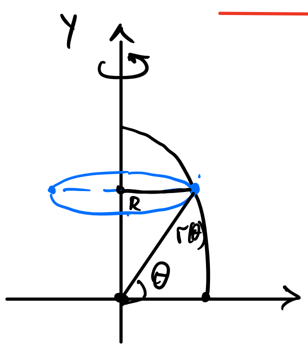

$$r(\theta) = 1+\sin\theta, \quad 0 \le \theta \le \frac{\pi}{2}$$

$$dA = TP_{väg} \cdot ds = 2\pi R \cdot ds$$

$$R = r(\theta)\cos\theta = (1+\sin\theta)\cos\theta$$

$$ds = \sqrt{r^2+r'^2}\,d\theta = \sqrt{(1+\sin\theta)^2+\cos^2\theta}\,d\theta = \sqrt{1+2\sin\theta+\underbrace{\sin^2\theta+\cos^2\theta}_{=1}}\,d\theta =$$

$$= \sqrt{2+2\sin\theta}\,d\theta = \sqrt{2}\sqrt{1+\sin\theta}\,d\theta. \quad \Rightarrow$$

$$dA = 2\pi(1+\sin\theta)\cos\theta \cdot \sqrt{2}\ \sqrt{1+\sin\theta}\ d\theta = 2\pi\sqrt{2}\ (1+\sin\theta)^{3/2}\cos\theta d\theta$$

$$\Rightarrow \quad A = \int_0^{\pi/2} dA = 2\pi\sqrt{2}\int_0^{\pi/2} (1+\sin\theta)^{3/2} \cos\theta\, d\theta = \left/ \begin{array}{l} t = 1+\sin\theta \\ dt = \cos\theta\, d\theta \\ \theta = 0 \Rightarrow \alpha = 1 \\ \theta = \pi/2 \Rightarrow \beta = 2 \end{array} \right/ =$$

$$= 2\pi\sqrt{2} \int_1^2 t^{3/2}\, dt = 2\pi\sqrt{2} \left(\frac{t^{5/2}}{5/2}\right)_1^2 =$$

$$= 2\pi\sqrt{2} \cdot \frac{2}{5} \left(2^{5/2} - 1\right) = \frac{4\sqrt{2}}{5}\pi\left(4\sqrt{2} - 1\right) \quad (a.e.)$$

Svar: $A = \frac{4\sqrt{2}}{5}\pi\left(4\sqrt{2} - 1\right)$ a.e.