Lösningsskiss till tentamen i Matematisk analys

764G07, del 2. 2021-03-15

1a $\int (2x+3)e^{3x}dx = \Big/ \begin{array}{l} \text{PI} \\ g=2x+3 \Rightarrow g'=2 \\ f=e^{3x} \Rightarrow F=\frac{1}{3}e^{3x} \end{array} \Big/ = (2x+3)\frac{e^{3x}}{3} - \int 2\cdot\frac{1}{3}e^{3x}dx =$

$= \frac{2x+3}{3}e^{3x} - \frac{2}{9}e^{3x} + C = \frac{6x+7}{9}e^{3x} + C$

1b $\int \frac{x}{x^2+4}dx \Big/ \begin{array}{l} t=x^2+4 \\ dt=2x\,dx \end{array} \Big/ = \frac{1}{2}\int\frac{dt}{t} = \frac{1}{2}\ln|t|+C = \frac{1}{2}\ln(x^2+4)+C$

1c $\int \frac{\cos x}{\sin^2 x + 2\sin x}dx = \Big/ \begin{array}{l} t=\sin x \\ dt=\cos x\,dx \end{array} \Big/ = \int\frac{1}{t^2+2t}dt = \int\frac{1}{t(t+2)}dt$

$\Big/ \frac{1}{t(t+2)} = \frac{1/2}{t} - \frac{1/2}{t+2} \Big/ = \frac{1}{2}\int\frac{dt}{t} - \frac{1}{2}\int\frac{dt}{t+2} = \frac{1}{2}\ln|t| - \frac{1}{2}\ln|t+2| +$

$+ C = \frac{1}{2}\ln\left|\frac{t}{t+2}\right| + C = \frac{1}{2}\ln\left|\frac{\sin x}{\sin x+2}\right| + C$

Svar: a) $\frac{6x+7}{9}e^{3x}+C$, b) $\frac{1}{2}\ln(x^2+4)+C$, c) $\frac{1}{2}\ln\left|\frac{\sin x}{\sin x+2}\right|+C$

2 $y' + y = \frac{1}{e^x+1}$ - linjär diffekvation av ordning 1

med $f(x)=1 \Rightarrow F(x)=x \Rightarrow IF = e^x \Rightarrow$

$e^x y' + e^x y = \frac{e^x}{e^x+1} \Leftrightarrow (e^x y)' = \frac{e^x}{e^x+1} \Rightarrow$

$e^x y = \int\frac{e^x}{e^x+1}dx = \Big/ \begin{array}{l} t=e^x+1 \\ dt=e^x dx \end{array} \Big/ = \int\frac{dt}{t} = \ln|t| + C =$

$= \ln|e^x+1| + C \Rightarrow y = e^{-x}\ln|e^x+1| + Ce^{-x}$

$y(0)=0 \Rightarrow 0 = \ln 2 + C \Rightarrow C = -\ln 2 \Rightarrow y = e^{-x}\ln(e^x+1) - e^{-x}\ln 2$

$\Rightarrow y = e^{-x}\ln\frac{e^x+1}{2}.$

Svar: $y = e^{-x}\ln\frac{e^x+1}{2}.$

3a $\lim_{x\to 0}\frac{\sin(2x)-2x}{x^3} = \lim_{x\to 0}\frac{2x - \frac{1}{3!}(2x)^3 + O(x^5) - 2x}{x^3} =$

$$= \lim_{x\to 0} \frac{-\frac{4}{3}x^3 + O(x^5)}{x^3} = \lim_{x\to 0}\left(-\frac{4}{3} + O(x^2)\right) = -\frac{4}{3}.$$

3b

$$\lim_{x\to 0} \frac{\ln(1+x^2)}{1-\cos x} = \lim_{x\to 0} \frac{x^2 + O(x^4)}{1-\left(1-\frac{x^2}{2}+O(x^4)\right)} = \lim_{x\to 0} \frac{x^2(1+O(x^2))}{x^2\left(\frac{1}{2}+O(x^2)\right)}$$

$$= \lim_{x\to 0} \frac{1+O(x^2)}{\frac{1}{2}+O(x^2)} = 2$$

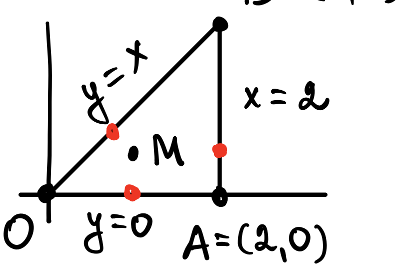
3c

$$\lim_{x\to 1} (2x-1)^{\frac{1}{x-1}} \Bigg/ \begin{matrix} t = x-1 \\ t\to 0 \text{ då } x\to 1 \\ x = t+1 \end{matrix} \Bigg/ = \lim_{t\to 0} (2t+1)^{\frac{1}{t}} =$$

$$= \lim_{t\to 0} e^{\frac{\ln(1+2t)}{t}} \Bigg/ \lim_{t\to 0} \frac{\ln(1+2t)}{t} = \lim_{t\to 0} \frac{2t+O(t^2)}{t} = \lim_{t\to 0}(2+O(t)) = 2 \Bigg/ = e^2.$$

Svar: a) $-\frac{4}{3}$ b) 2 c) e^2.

4



- $f(x,y) = x^2 - 2x + y^2 - y + 2$ - kontinuerlig

$\mathcal{D} = \{(x,y):\ 0 \le y \le x \le 2\}$ - kompakt område (slutet och begränsat)

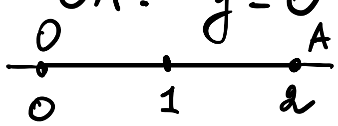
$\Rightarrow$ f antar f_{max} och f_{min} på området $\mathcal{D}$.

- stationära punkter:

$$\begin{cases} f'_x = 2x-2 = 0 \\ f'_y = 2y-1 = 0 \end{cases} \Rightarrow \begin{cases} x = 1 \\ y = 1/2 \end{cases} \Rightarrow M = \left(1, \tfrac{1}{2}\right) \Rightarrow f\left(1,\tfrac{1}{2}\right) = \frac{3}{4}$$

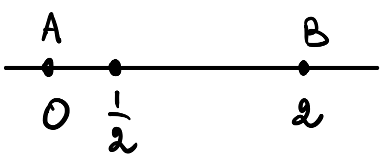
- randpunkter:

– OA: $y = 0 \Rightarrow f(x,0) = x^2 - 2x + 2, \quad 0 \le x \le 2$



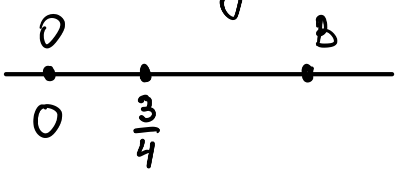
$f'(x,0) = 2x - 2 = 0 \Rightarrow x = 1 \Rightarrow f(1,0) = 1$

– AB: $x = 2 \Rightarrow f(2,y) = y^2 - y + 2, \quad 0 \le y \le 2$



$f'(2,y) = 2y - 1 = 0 \Rightarrow y = \frac{1}{2} \Rightarrow f\left(2,\tfrac{1}{2}\right) = \frac{7}{4}$

$-\ OB:\ y = x \Rightarrow f(x,x) = 2x^2 - 3x + 2, \quad 0 \le x \le 2$



$$f'(x,x) = 4x - 3 = 0 \Rightarrow x = \frac{3}{4} \Rightarrow f\left(\frac{3}{4},\frac{3}{4}\right) = \frac{7}{8}$$

$$f(0,0) = 2, \quad f(2,2) = f(0,2) = 4$$

Svar: $f_{max} = f(2,2) = 4, \quad f_{min} = f\left(1, \frac{1}{2}\right) = \frac{3}{4}.$

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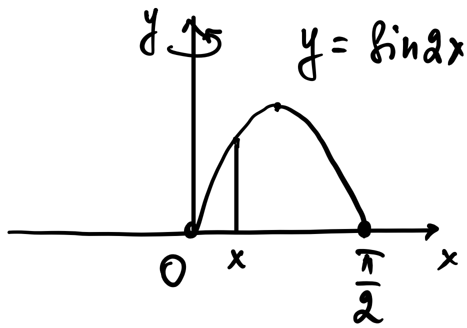

$$D = \left\{ (x,y):\ 0 \le x \le \frac{\pi}{2},\ 0 \le y \le \sin 2x \right\}$$

$$dV = 2\pi x\, f(x)\, dx$$

$$V = \int_0^{\pi/2} 2\pi x \cdot \sin(2x)\, dx = 2\pi \int_0^{\pi/2} x \sin(2x)\, dx$$

$$= \left/ \begin{array}{l} PI \\ g = x \Rightarrow g' = 1 \\ f = \sin(2x) \Rightarrow F = -\frac{1}{2}\cos(2x) \end{array} \right/ = 2\pi \left(\left(-\frac{x}{2}\cos(2x) \right)_0^{\pi/2} + \frac{1}{2}\int_0^{\pi/2} \cos(2x)\, dx \right)$$

$$= 2\pi \left(-\frac{\pi}{4} \cdot (-1) + \frac{1}{4}\left(\sin 2x \right)_0^{\pi/2} \right) = 2\pi \cdot \frac{\pi}{4} + 0 - 0 = \frac{\pi^2}{2} \ (v.e.)$$

Svar: $V = \frac{\pi^2}{2}$ v.e.

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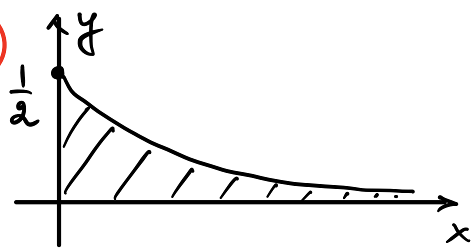

$$y = \frac{1}{x^2 + 3x + 2}, \quad x \ge 0$$

$$A = \int_0^{\infty} \frac{1}{x^2 + 3x + 2}\, dx = \lim_{b \to \infty} \int_0^{b} \frac{dx}{(x+1)(x+2)}$$

$$\left/ \frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2} \right/ = \lim_{b \to \infty} \int_0^{b} \left(\frac{1}{x+1} - \frac{1}{x+2} \right) dx =$$

$$= \lim_{b \to \infty} \left[\ln|x+1| - \ln|x+2| \right]_0^{b} = \lim_{b \to \infty} \left[\ln\left| \frac{x+1}{x+2} \right| \right]_0^{b} =$$

$$= \lim_{b \to \infty} \left(\ln \underbrace{\frac{b+1}{b+2}}_{\to 1} - \ln\frac{1}{2} \right) = -\ln\frac{1}{2} = \ln 2 \ (a.e.)$$

Svar: $A = \ln 2$ a.e.

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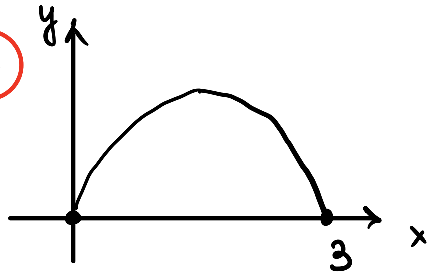

$$y = \sqrt{x}\left(1 - \frac{x}{3}\right), \quad 0 \le x \le 3$$

$$ds = \sqrt{1 + f'^2(x)}\, dx$$

$$f(x) = x^{1/2} - \frac{1}{3}x^{3/2}; \quad f' = \frac{1}{2}x^{-1/2} - \frac{1}{2}x^{1/2} =$$

$$= \frac{1}{2}\frac{1-x}{\sqrt{x}} \Rightarrow \quad 1 + f'^2 = 1 + \frac{1}{4}\frac{(1-x)^2}{x} = \frac{4x + 1 - 2x + x^2}{4x} = \frac{1 + 2x + x^2}{4x}$$

$$\Rightarrow ds = \sqrt{\frac{(1+x)^2}{4x}}\, dx = \frac{1+x}{2\sqrt{x}}\, dx \Rightarrow$$

$$s = \int_0^3 ds = \int_0^3 \frac{1+x}{2\sqrt{x}}\, dx = \frac{1}{2}\int_0^3 \left(\frac{1}{\sqrt{x}} + \sqrt{x}\right) dx = \frac{1}{2}\left(2\sqrt{x} + \frac{x^{3/2}}{3/2}\right)\Bigg|_0^3 =$$

$$= \frac{1}{2}\left(2\sqrt{3} + \frac{2}{3}\cdot 3\sqrt{3}\right) = 2\sqrt{3} \quad (\text{l.e.})$$

Svar: $s = 2\sqrt{3}$ l.e.