Lösningsskiss till tentamen i Matematisk analys, del 2
2021-08-21

1a

$$\int_0^4 x\sqrt{9+x^2}\,dx = \left/ \begin{array}{l} t = 9+x^2 \\ dt = 2x\,dx \\ x=0 \Rightarrow \alpha = 9 \\ x=4 \Rightarrow \beta = 25 \end{array} \right/ = \int_9^{25} \sqrt{t}\ \tfrac{1}{2}\,dt = \left[\tfrac{1}{2}\cdot\tfrac{2}{3}\,t^{3/2}\right]_9^{25} =$$

$$= \tfrac{1}{3}\,25\sqrt{25} - \tfrac{1}{3}\,9\cdot\sqrt{9} = \tfrac{1}{3}(125-27) = \tfrac{98}{3}.$$

Svar: $\frac{98}{3}$.

1b

$$\int x\sin(3x)\,dx = \left/ \begin{array}{l} \text{PI} \\ g = x \Rightarrow g' = 1 \\ f = \sin 3x \Rightarrow F = -\tfrac{1}{3}\cos 3x \end{array} \right/ = -\tfrac{1}{3}x\cos 3x + \tfrac{1}{3}\int \cos 3x\,dx$$

$$= -\tfrac{1}{3}x\cos 3x + \tfrac{1}{9}\sin 3x + C.$$

Svar: $-\frac{1}{3}x\cos 3x + \frac{1}{9}\sin 3x + C$.

1c

$$\int \frac{1}{x^3-x}\,dx = \int \frac{1}{x(x-1)(x+1)}\,dx = \left/ \frac{1}{x(x-1)(x+1)} = \frac{-1}{x} + \frac{1/2}{x-1} + \frac{1/2}{x+1} \right/ =$$

$$= -\int\frac{dx}{x} + \tfrac{1}{2}\int\frac{dx}{x-1} + \tfrac{1}{2}\int\frac{dx}{x+1} = -\ln|x| + \tfrac{1}{2}\ln|x-1| + \tfrac{1}{2}\ln|x+1| + C$$

$$= \tfrac{1}{2}\ln\frac{|x^2-1|}{x^2} + C.$$

Svar: $\frac{1}{2}\ln\frac{|x^2-1|}{x^2} + C$.

2

$xy' + 2y = \frac{x}{x-1},\quad x>1,\quad$ - lin. DE

- $y' + \frac{2}{x}y = \frac{1}{x-1},\quad x>1$
- $F(x) = \int \frac{2}{x}\,dx = 2\ln x + C \Rightarrow IF = e^{\ln x^2} = x^2.$
- $x^2y' + 2xy = \frac{x^2}{x-1} \Rightarrow (x^2y)' = \frac{x^2}{x-1} \Rightarrow$

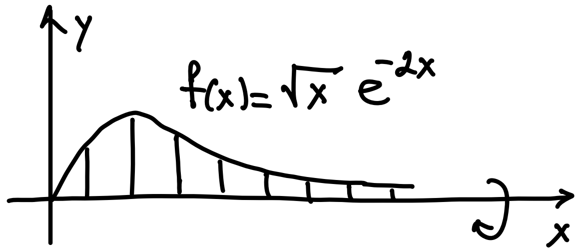
$$x^2y = \int\frac{x^2}{x-1}\,dx = \int\left(x+1+\frac{1}{x-1}\right)dx = \tfrac{1}{2}x^2 + x + \ln(x-1) + C$$

$$\Rightarrow y = \tfrac{1}{2} + \tfrac{1}{x} + \tfrac{1}{x^2}\ln(x-1) + \tfrac{C}{x^2},\quad x>1.$$

- $y(2) = 1 \Rightarrow 1 = \frac{1}{2} + \frac{1}{2} + \frac{1}{4}\ln 1 + \frac{C}{4} \Rightarrow C = 0$

Sv $\quad y = \frac{1}{2} + \frac{1}{x} + \frac{1}{x^2}\ln(x-1).$

3.



$$dV = \pi f^2(x)\,dx = \pi x e^{-4x}\,dx$$

$$V = \pi\int_0^{\infty} x e^{-4x}\,dx = \pi \lim_{b\to\infty}\int_0^{b} x e^{-4x}\,dx =$$

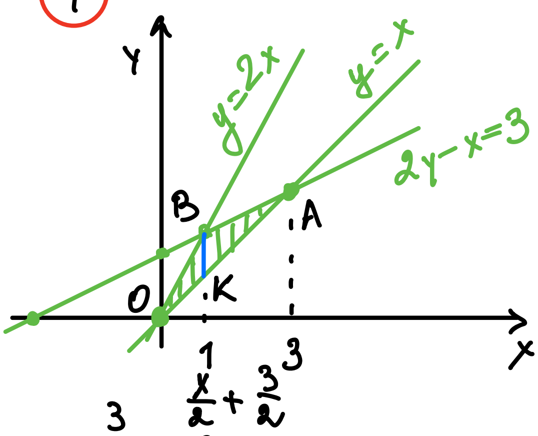
$$\Big/ \overset{PI}{\begin{matrix} g=x \Rightarrow g'=1 \\ f=e^{-4x} \Rightarrow F=-\frac{1}{4}e^{-4x}\end{matrix}} \Big/ = \pi \lim_{b\to\infty}\left[\left(-\frac{x}{4}e^{-4x}\right)_0^b + \frac{1}{4}\int_0^b e^{-4x}\,dx\right] =$$

$$= \pi \lim_{b\to\infty}\left[-\frac{1}{4}\frac{b}{e^{4b}} - \frac{1}{4}\cdot\frac{1}{4}\left(e^{-4x}\right)_0^b\right] =$$

$$= \pi \lim_{b\to\infty}\left[-\frac{1}{4}\underbrace{\frac{b}{e^{4b}}}_{\to 0} - \frac{1}{16}\underbrace{e^{-4b}}_{\to 0} + \frac{1}{16}\right] = \frac{\pi}{16} \quad (\text{v.e.})$$

Svar: $V = \frac{\pi}{16}$ v.e.

4



$\mathcal{D} = \triangle OAB \Rightarrow O=(0,0),\ A=(3,3),\ B=(1,2)$

$$\iint_{\mathcal{D}} (2x+2y)\,dx\,dy = \iint_{\triangle OKB} (2x+2y)\,dx\,dy + \iint_{\triangle KAB} (2x+2y)\,dx\,dy = \int_0^1\left(\int_x^{2x}(2x+2y)\,dy\right)dx +$$

$$+\int_1^3\left(\int_x^{\frac{x}{2}+\frac{3}{2}}(2x+2y)\,dy\right)dx = \int_0^1\left(2xy+y^2\right)_{y=x}^{y=2x}dx +$$

$$+\int_1^3\left(2xy+y^2\right)_{y=x}^{y=\frac{x+3}{2}}dx = \int_0^1\left(2x(2x-x)+4x^2-x^2\right)dx +$$

$$+\int_1^3\left(x(x+3-2x)+\frac{(x+3)^2}{4}-x^2\right)dx = \int_0^1 5x^2\,dx +$$

$$+\int_1^3\left(3x-x^2+\frac{1}{4}(x^2+6x+9)-x^2\right)dx = \frac{5}{3}\left(x^3\right)_0^1 + \left(\frac{9}{4}x^2-\frac{7}{12}x^3+\frac{9}{4}x\right)_1^3 =$$

$$= \frac{5}{3} + \frac{9}{4}\cdot 8 - \frac{7}{\cancel{12}_{6}}\cdot \cancel{26}^{13} + \frac{9}{4}\cdot 2 = \frac{5}{3} + \frac{\cancel{90}^{45}}{\cancel{4}_{2}} - \frac{91}{6} = \frac{10+135-91}{6} = \frac{54}{6} =$$

$$= 9.$$

Svar: 9.

5a

$$\lim_{x\to 0} \frac{e^{x^2+x^4} - 1 - x^2}{\cos(2x^2)-1} = \Big/ \begin{array}{l} \cos t - 1 = 1 - \frac{t^2}{2} + O(t^4) - 1 \\ e^t = 1 + t + \frac{t^2}{2} + O(t^3) \end{array} \Big/$$

$$= \lim_{x\to 0} \frac{1 + (x^2+x^4) + \frac{1}{2}(x^2+x^4)^2 + O(x^6) - 1 - x^2}{-\frac{(2x^2)^2}{2} + O(x^8)} =$$

$$= \lim_{x\to 0} \frac{x^4 + \frac{1}{2}x^4 + O(x^6) + O(x^6)}{-2x^4 + O(x^8)} = \lim_{x\to 0} \frac{\frac{3}{2}x^4 + O(x^6)}{-2x^4+O(x^8)} =$$

$$= \lim_{x\to 0} \frac{\cancel{x^4}\left(\frac{3}{2} + O(x^2)\right)}{\cancel{x^4}\left(-2 + O(x^4)\right)} = \frac{3/2}{-2} = -\frac{3}{4}$$

Svar: $-\frac{3}{4}$.

5b

$$\lim_{x\to 0} \frac{2\ln(1+x) + \sin x + ax^2 - 3x}{x^3} =$$

$$= \lim_{x\to 0} \frac{2\left(x - \frac{x^2}{2} + \frac{x^3}{3} + O(x^4)\right) + x - \frac{x^3}{3!} + O(x^5) + ax^2 - 3x}{x^3} =$$

$$= \lim_{x\to 0} \frac{x^2(a-1) + x^3\left(\frac{2}{3} - \frac{1}{3!}\right) + O(x^4)}{x^3} = A \text{ är ändligt om}$$

och endast om $a = 1$. I så fall

$$A = \lim_{x\to 0} \frac{\frac{1}{2}x^3 + O(x^4)}{x^3} = \lim_{x\to 0} \frac{x^3\left(\frac{1}{2} + O(x)\right)}{x^3} = \frac{1}{2}.$$

Svar: $a = 1$; $A = \frac{1}{2}$.

6. $f(x) = x^3\left(a e^{x^2} - \cos x^3\right) = x^3\left(a\left(1+x^2+\frac{x^4}{2}+\frac{x^6}{3!}+O(x^8)\right)\right.$

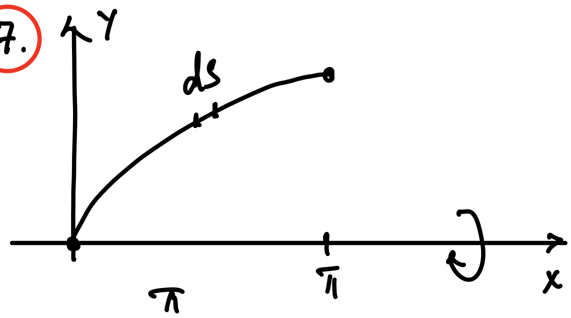
$\left. - \left(1-\frac{x^6}{2}+O(x^{12})\right)\right) = ax^3+ax^5+\frac{a}{2}x^7+\frac{a}{3!}x^9+O(x^{11}) -$

$- x^3+\frac{x^9}{2}+O(x^{15}) = (a-1)x^3+ax^5+\frac{a}{2}x^7+\left(\frac{a}{3!}+\frac{1}{2}\right)x^9+O(x^{11})$

Notera att $f(x) = f(0)+f'(0)x+\frac{f''(0)}{2}x^2+\dots+\frac{f^{(9)}(0)}{9!}x^9+O(x^{10})$

$\Rightarrow f^{(9)}(0) = 0 \Leftrightarrow \frac{a}{3!}+\frac{1\cdot 3}{2\cdot 3}=0 \Leftrightarrow a=-3$

Svar: $a=-3$.

7. 

$\begin{cases} x = t-\sin t \\ y = 1-\cos t \end{cases} \quad 0 \le t \le \pi$

$dA = 2\pi y \cdot ds = 2\pi y\sqrt{x'^2+y'^2}\,dt$

$A = 2\pi\int_0^\pi (1-\cos t)\sqrt{(1-\cos t)^2+\sin^2 t}\,dt =$

$= 2\pi\int_0^\pi (1-\cos t)\sqrt{2-2\cos t}\,dt = 2\sqrt{2}\pi\int_0^\pi (1-\cos t)^{3/2}dt =$

$= 2\sqrt{2}\pi\int_0^\pi \left(2\sin^2\frac{t}{2}\right)^{3/2}dt = 2^3\pi\int_0^\pi \sin^3\frac{t}{2}\,dt =$

$= 8\pi\int_0^\pi \left(1-\cos^2\frac{t}{2}\right)\sin\frac{t}{2}\,dt = \left/ \begin{array}{l} u=\cos\frac{t}{2} \\ du = -\frac{1}{2}\sin\frac{t}{2}\,dt \Rightarrow \sin\frac{t}{2}dt = -2du \\ t=0 \Rightarrow \alpha=1 \\ t=\pi \Rightarrow \beta=0 \end{array} \right/ =$

$= 8\pi\int_1^0 (1-u^2)(-2)du = 16\pi\int_0^1 (1-u^2)\,du = 16\pi\left(u-\frac{u^3}{3}\right)_0^1 =$

$= 16\pi\left(1-\frac{1}{3}\right) = \frac{32}{3}\pi$ (a.e.)

Svar: $A = \frac{32}{3}\pi$ a.e.