Lösningsskiss till tentamen i Matematisk analys, del 2,

764G07, TEN2, 2019-08-24.

1a $\int x \ln x \, dx \;\Big/\; \begin{matrix} g = \ln x & g' = \frac{1}{x} \\ f = x & F = \frac{1}{2}x^2 \end{matrix} \Big/ = \frac{1}{2}x^2 \ln x - \frac{1}{2}\int x^2 \frac{1}{x}\,dx =$

$= \underline{\frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C}$

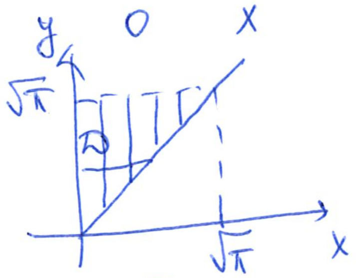
1b $\int \frac{3x+7}{x^2+3x+2}\,dx \;\Big/\; \frac{3x+7}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2} \Rightarrow \begin{matrix} A = 4 \\ B = 1 \end{matrix} \Big/ =$

$= \int \left(\frac{4}{x+1} - \frac{1}{x+2}\right) dx = \underline{4 \ln|x+1| - \ln|x+2| + C}$

1c $\int \sin x \cos x \, dx \;\Big/\; \begin{matrix} t = \sin x \\ dt = \cos x \, dx \end{matrix} \Big/ = \int t \, dt = \frac{1}{2}t^2 + C =$

$= \underline{\frac{1}{2}\sin^2 x + C}$

2 $I = \int_0^{\sqrt{\pi}} \left(\int_x^{\sqrt{\pi}} \sin(y^2)\,dy \right) dx \qquad D = \left\{ (x,y) : \begin{matrix} 0 \le x \le \sqrt{\pi} \\ x \le y \le \sqrt{\pi} \end{matrix} \right\}$



$I = \iint_D \sin(y^2)\,dy\,dx = \int_0^{\sqrt{\pi}} \left(\int_0^{y} \sin(y^2)\,dx \right) dy$

$= \int_0^{\sqrt{\pi}} \sin(y^2) \cdot y \, dy = \Big/ \begin{matrix} t = y^2 \\ dt = 2y\,dy \end{matrix} \Big/ =$

$= \frac{1}{2}\int_0^{\pi} \sin t \, dt = \left(-\frac{1}{2}\cos t\right)_0^{\pi} = \frac{1}{2} \cdot 2 = 1.$

<u>Svar:</u> $I = 1$

3a $\lim_{x \to 0} \frac{\sin x - \ln(1+x)}{x \arctan x} = \lim_{x \to 0} \frac{x + O(x^3) - (x - \frac{x^2}{2} + O(x^3))}{x(x + O(x^3))}$

$= \lim_{x \to 0} \frac{x^2(\frac{1}{2} + O(x))}{x^2(1 + O(x))} = \frac{1}{2}$

3b $\lim_{x \to 0} \frac{e^x - \cos x + \ln(1+ax)}{x^2} =$

$= \lim_{x \to 0} \frac{\cancel{1} + x + \frac{x^2}{2} + O(x^3) - (\cancel{1} - \frac{x^2}{2} + O(x^4)) + (ax - \frac{a^2x^2}{2} + O(x^3))}{x^2}$

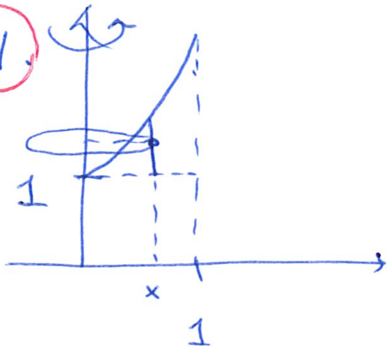
$= \lim_{x \to 0} \frac{(1+a)x + \frac{x^2}{2}(1-a^2) + O(x^3)}{x^2}$ är ändligt $\Leftrightarrow$

$1 + a = 0 \Leftrightarrow a = -1$

Om $a=-1$, får vi $\lim\limits_{x\to 0} \dfrac{\frac{1}{2}x^2+O(x^3)}{x^2} = \lim\limits_{x\to 0}\left(\frac{1}{2}+O(x)\right) = \frac{1}{2}$.

Svar a) $\frac{1}{2}$. b) $a=-1$, gränsvärdet är $\frac{1}{2}$.

4.



$1 \le y \le e^x$, $\quad 0 \le x \le 1$.

$dV = 2\pi x \cdot (e^x-1)\,dx \Rightarrow V = \int_0^1 dV =$

$= 2\pi\int_0^1 x(e^x-1)\,dx = 2\pi\left(\int_0^1 x e^x\,dx - \int_0^1 x\,dx\right) =$

$= 2\pi\left(xe^x - e^x - \frac{x^2}{2}\right)\Big|_0^1 = 2\pi\left(-\frac{1}{2}+1\right) = \pi$ (v.e.)

Svar: $V = \pi$ v.e.

5. $\begin{cases} y' + \frac{1}{x}y = x\cos x, & x>0 \\ y(\pi)=0 \end{cases}$ $\quad$ • $IF = e^{\int \frac{1}{x}dx} = x$

$\Downarrow$

• $xy' + y = x^2\cos x \Leftrightarrow$

$(xy)' = x^2\cos x \Rightarrow$

$xy = \int x^2\cos x\,dx = + x^2\sin x - \int 2x\sin x\,dx = x^2\sin x - \left(-2x\cos x + \int 2\cos x\,dx\right) =$

$= x^2\sin x + 2x\cos x - 2\sin x + C = (x^2-2)\sin x + 2x\cos x + C$

$\Rightarrow \quad y = \dfrac{x^2-2}{x}\sin x + 2\cos x + \dfrac{C}{x}$

• $y(\pi)=0 \Rightarrow 0 = -2 + \frac{C}{\pi} \Rightarrow C = 2\pi$

Svar: $y = \dfrac{x^2-2}{x}\sin x + 2\cos x + \dfrac{2\pi}{x}$.

6. $\begin{cases} x(\theta) = 2\cos\theta - \cos 2\theta \\ y(\theta) = 2\sin\theta - \sin 2\theta \end{cases}$ $\quad 0 \le \theta \le 2\pi$. $\quad ds = \sqrt{x'^2+y'^2}\,d\theta$

$\begin{cases} x' = -2\sin\theta + 2\sin 2\theta \\ y' = 2\cos\theta - 2\cos 2\theta \end{cases}$

$ds = \sqrt{(-2\sin\theta + 2\sin 2\theta)^2 + (2\cos\theta - 2\cos 2\theta)^2}\,d\theta$

$= \sqrt{8 - 8(\sin\theta\sin 2\theta + \cos\theta\cos 2\theta)}\,d\theta$

$\sqrt{8 - 8\cos\theta}\,d\theta = \sqrt{16\sin^2\frac{\theta}{2}}\,d\theta = 4\left|\sin\frac{\theta}{2}\right|d\theta$

$$s = \int_0^{2\pi} 4\left|\sin\frac{\theta}{2}\right| d\theta \Bigg/ \begin{matrix} t = \frac{\theta}{2} & \theta = 0 \Rightarrow \alpha = 0 \\ dt = \frac{1}{2}d\theta & \theta = 2\pi \Rightarrow \beta = \pi \end{matrix} \Bigg/ = 4\cdot 2\int_0^{\pi} \sin t\,dt =$$

$$= 8\left(-\cos t\right)_0^{\pi} = 16 \text{ (l.e.)}$$

Svar: $s = 16$ l.e.

7. $\begin{cases} e^x y'' - x^2 y = 0 & (1) \\ y(0) = 1,\ y'(0) = 0 \end{cases}$

- sätter in $x = 0$ i (1) $\Rightarrow$ $y''(0) = 0$
- Vi deriverar (1):

$$e^x y'' + e^x y''' - 2xy - x^2 y' = 0 \qquad (2)$$

- sätter in $x = 0$ $\Rightarrow$ $y''(0) + y'''(0) - 0 - 0 = 0$

Eftersom $y''(0) = 0$ får vi $y'''(0) = 0$

- Vi deriverar (2): $e^x y'' + e^x y''' + e^x y''' + e^x y^{(4)} - 2y -$

$$- 2xy' - 2xy' - 2xy'' = 0. \qquad (3)$$

- Sätter in $x = 0$ i (3) $\Rightarrow$ $y''(0) + y'''(0) + y'''(0) + y^{(4)}(0)$

$$- 2y(0) - 0 - 0 - 0 = 0$$

- Eftersom $y''(0) = y'''(0) = 0$ och $y(0) = 1$ får vi

$y^{(4)}(0) - 2 = 0$ dvs $y^4(0) = 2$.

- Från det följer att Maclaurinutvecklingen för y är

$$y(x) = y(0) + y'(0)x + \frac{y''(0)}{2}x^2 + \frac{y'''(0)}{3!}x^3 + \frac{y^{(4)}(0)}{4!}x^4 + r(x) =$$

$$= 1 + \frac{2}{4!}x^4 + r(x)$$

- Alltså $y(x) - y(0) = \frac{1}{12}x^4 + r(x) > 0$ för alla x nära 0 dvs $y(x) > y(0)$ i en omgivning av origo.

Det betyder att y har lokalt minimum för $x = 0$

Svar: $y(x)$ har lokalt minimum i origo.