The Gram–Schmidt process

We define the projection operator by

$$\mathrm{proj}_{\mathbf{u}}(\mathbf{v}) = \frac{\langle \mathbf{v}, \mathbf{u} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle}\mathbf{u},$$

where ‹ **v**,**u** › denotes the inner product of the vectors **v** and **u**. This operator projects the vector **v** orthogonally onto the line spanned by vector **u**.

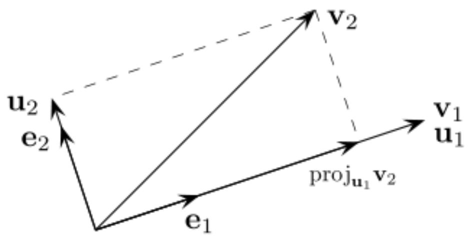
The Gram–Schmidt process then works as follows:

$$\begin{aligned}
\mathbf{u}_1 &= \mathbf{v}_1, & \mathbf{e}_1 &= \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|} \\
\mathbf{u}_2 &= \mathbf{v}_2 - \mathrm{proj}_{\mathbf{u}_1}(\mathbf{v}_2), & \mathbf{e}_2 &= \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|} \\
\mathbf{u}_3 &= \mathbf{v}_3 - \mathrm{proj}_{\mathbf{u}_1}(\mathbf{v}_3) - \mathrm{proj}_{\mathbf{u}_2}(\mathbf{v}_3), & \mathbf{e}_3 &= \frac{\mathbf{u}_3}{\|\mathbf{u}_3\|} \\
\mathbf{u}_4 &= \mathbf{v}_4 - \mathrm{proj}_{\mathbf{u}_1}(\mathbf{v}_4) - \mathrm{proj}_{\mathbf{u}_2}(\mathbf{v}_4) - \mathrm{proj}_{\mathbf{u}_3}(\mathbf{v}_4), & \mathbf{e}_4 &= \frac{\mathbf{u}_4}{\|\mathbf{u}_4\|} \\
&\vdots & &\vdots \\
\mathbf{u}_k &= \mathbf{v}_k - \sum_{j=1}^{k-1} \mathrm{proj}_{\mathbf{u}_j}(\mathbf{v}_k), & \mathbf{e}_k &= \frac{\mathbf{u}_k}{\|\mathbf{u}_k\|}.
\end{aligned}$$

Example

Consider the following set of vectors in $\mathbf{R}^2$ (with the conventional inner product)

$$S = \left\{ \mathbf{v}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \right\}.$$



Now, perform Gram–Schmidt, to obtain an orthogonal set of vectors:

$$\mathbf{u}_1 = \mathbf{v}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$\mathbf{u}_2 = \mathbf{v}_2 - \mathrm{proj}_{\mathbf{u}_1}(\mathbf{v}_2) = \begin{pmatrix} 2 \\ 2 \end{pmatrix} - \mathrm{proj}_{\left(\begin{smallmatrix} 3 \\ 1 \end{smallmatrix}\right)} \left(\begin{pmatrix} 2 \\ 2 \end{pmatrix} \right) = \begin{pmatrix} -2/5 \\ 6/5 \end{pmatrix}.$$

We check that the vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ are indeed orthogonal:

$$\langle \mathbf{u}_1, \mathbf{u}_2 \rangle = \left\langle \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \begin{pmatrix} -2/5 \\ 6/5 \end{pmatrix} \right\rangle = -\frac{6}{5} + \frac{6}{5} = 0,$$

noting that if the dot product of two vectors is *0* then they are orthogonal.

We can then normalize the vectors by dividing out their sizes as shown above:

$$\mathbf{e}_1 = \frac{1}{\sqrt{10}} \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$\mathbf{e}_2 = \frac{1}{\sqrt{\frac{40}{25}}} \begin{pmatrix} -2/5 \\ 6/5 \end{pmatrix} = \frac{1}{\sqrt{10}} \begin{pmatrix} -1 \\ 3 \end{pmatrix}.$$

Problems

Perform the Gram-Schmidt process on each of these sets of vectors.

1. $\left\langle \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\rangle$ 2. $\left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \end{pmatrix} \right\rangle$ 3. $\left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right\rangle$ 4. $\left\langle \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} \right\rangle$ 5. $\left\langle \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \right\rangle$

Answers

1. $\vec{\kappa}_1 = \begin{pmatrix}1\\1\end{pmatrix} \qquad \vec{\kappa}_2 = \begin{pmatrix}2\\1\end{pmatrix} - \text{proj}_{[\vec{\kappa}_1]}(\begin{pmatrix}2\\1\end{pmatrix}) = \begin{pmatrix}2\\1\end{pmatrix} - \frac{\begin{pmatrix}2\\1\end{pmatrix}\cdot\begin{pmatrix}1\\1\end{pmatrix}}{\begin{pmatrix}1\\1\end{pmatrix}\cdot\begin{pmatrix}1\\1\end{pmatrix}}\cdot\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix} - \frac{3}{2}\cdot\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1/2\\-1/2\end{pmatrix}$

2. $\vec{\kappa}_1 = \begin{pmatrix}0\\1\end{pmatrix} \qquad \vec{\kappa}_2 = \begin{pmatrix}-1\\3\end{pmatrix} - \text{proj}_{[\vec{\kappa}_1]}(\begin{pmatrix}-1\\3\end{pmatrix}) = \begin{pmatrix}-1\\3\end{pmatrix} - \frac{\begin{pmatrix}-1\\3\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix}}{\begin{pmatrix}0\\1\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix}}\cdot\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}-1\\3\end{pmatrix} - \frac{3}{1}\cdot\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}-1\\0\end{pmatrix}$

3. $\vec{\kappa}_1 = \begin{pmatrix}0\\1\end{pmatrix} \qquad \vec{\kappa}_2 = \begin{pmatrix}-1\\0\end{pmatrix} - \text{proj}_{[\vec{\kappa}_1]}(\begin{pmatrix}-1\\0\end{pmatrix}) = \begin{pmatrix}-1\\0\end{pmatrix} - \frac{\begin{pmatrix}-1\\0\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix}}{\begin{pmatrix}0\\1\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix}}\cdot\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}-1\\0\end{pmatrix} - \frac{0}{1}\cdot\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}-1\\0\end{pmatrix}$

The corresponding orthonormal bases for the three parts of this question are these.

$$\langle\begin{pmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix},\begin{pmatrix}\sqrt{2}/2\\-\sqrt{2}/2\end{pmatrix}\rangle \qquad \langle\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}-1\\0\end{pmatrix}\rangle \qquad \langle\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}-1\\0\end{pmatrix}\rangle$$

4. $\vec{\kappa}_1 = \begin{pmatrix}2\\2\\2\end{pmatrix}$

$$\vec{\kappa}_2 = \begin{pmatrix}1\\0\\-1\end{pmatrix} - \text{proj}_{[\vec{\kappa}_1]}(\begin{pmatrix}1\\0\\-1\end{pmatrix}) = \begin{pmatrix}1\\0\\-1\end{pmatrix} - \frac{\begin{pmatrix}1\\0\\-1\end{pmatrix}\cdot\begin{pmatrix}2\\2\\2\end{pmatrix}}{\begin{pmatrix}2\\2\\2\end{pmatrix}\cdot\begin{pmatrix}2\\2\\2\end{pmatrix}}\cdot\begin{pmatrix}2\\2\\2\end{pmatrix} = \begin{pmatrix}1\\0\\-1\end{pmatrix} - \frac{0}{12}\cdot\begin{pmatrix}2\\2\\2\end{pmatrix} = \begin{pmatrix}1\\0\\-1\end{pmatrix}$$

$$\vec{\kappa}_3 = \begin{pmatrix}0\\3\\1\end{pmatrix} - \text{proj}_{[\vec{\kappa}_1]}(\begin{pmatrix}0\\3\\1\end{pmatrix}) - \text{proj}_{[\vec{\kappa}_2]}(\begin{pmatrix}0\\3\\1\end{pmatrix}) = \begin{pmatrix}0\\3\\1\end{pmatrix} - \frac{\begin{pmatrix}0\\3\\1\end{pmatrix}\cdot\begin{pmatrix}2\\2\\2\end{pmatrix}}{\begin{pmatrix}2\\2\\2\end{pmatrix}\cdot\begin{pmatrix}2\\2\\2\end{pmatrix}}\cdot\begin{pmatrix}2\\2\\2\end{pmatrix} - \frac{\begin{pmatrix}0\\3\\1\end{pmatrix}\cdot\begin{pmatrix}1\\0\\-1\end{pmatrix}}{\begin{pmatrix}1\\0\\-1\end{pmatrix}\cdot\begin{pmatrix}1\\0\\-1\end{pmatrix}}\cdot\begin{pmatrix}1\\0\\-1\end{pmatrix}$$

$$= \begin{pmatrix}0\\3\\1\end{pmatrix} - \frac{8}{12}\cdot\begin{pmatrix}2\\2\\2\end{pmatrix} - \frac{-1}{2}\cdot\begin{pmatrix}1\\0\\-1\end{pmatrix} = \begin{pmatrix}-5/6\\5/3\\-5/6\end{pmatrix}$$

5. $\vec{\kappa}_1 = \begin{pmatrix}1\\-1\\0\end{pmatrix}$

$$\vec{\kappa}_2 = \begin{pmatrix}0\\1\\0\end{pmatrix} - \text{proj}_{[\vec{\kappa}_1]}(\begin{pmatrix}0\\1\\0\end{pmatrix}) = \begin{pmatrix}0\\1\\0\end{pmatrix} - \frac{\begin{pmatrix}0\\1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\0\end{pmatrix}}{\begin{pmatrix}1\\-1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\0\end{pmatrix}}\cdot\begin{pmatrix}1\\-1\\0\end{pmatrix} = \begin{pmatrix}0\\1\\0\end{pmatrix} - \frac{-1}{2}\cdot\begin{pmatrix}1\\-1\\0\end{pmatrix} = \begin{pmatrix}1/2\\1/2\\0\end{pmatrix}$$

$$\vec{\kappa}_3 = \begin{pmatrix}2\\3\\1\end{pmatrix} - \text{proj}_{[\vec{\kappa}_1]}(\begin{pmatrix}2\\3\\1\end{pmatrix}) - \text{proj}_{[\vec{\kappa}_2]}(\begin{pmatrix}2\\3\\1\end{pmatrix})$$

$$= \begin{pmatrix}2\\3\\1\end{pmatrix} - \frac{\begin{pmatrix}2\\3\\1\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\0\end{pmatrix}}{\begin{pmatrix}1\\-1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\0\end{pmatrix}}\cdot\begin{pmatrix}1\\-1\\0\end{pmatrix} - \frac{\begin{pmatrix}2\\3\\1\end{pmatrix}\cdot\begin{pmatrix}1/2\\1/2\\0\end{pmatrix}}{\begin{pmatrix}1/2\\1/2\\0\end{pmatrix}\cdot\begin{pmatrix}1/2\\1/2\\0\end{pmatrix}}\cdot\begin{pmatrix}1/2\\1/2\\0\end{pmatrix}$$

$$= \begin{pmatrix}2\\3\\1\end{pmatrix} - \frac{-1}{2}\cdot\begin{pmatrix}1\\-1\\0\end{pmatrix} - \frac{5/2}{1/2}\cdot\begin{pmatrix}1/2\\1/2\\0\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

The corresponding orthonormal bases for the two parts of this question are these.

$$\langle\begin{pmatrix}1/\sqrt{3}\\1/\sqrt{3}\\1/\sqrt{3}\end{pmatrix},\begin{pmatrix}1/\sqrt{2}\\0\\-1/\sqrt{2}\end{pmatrix},\begin{pmatrix}-1/\sqrt{6}\\2/\sqrt{6}\\-1/\sqrt{6}\end{pmatrix}\rangle \qquad \langle\begin{pmatrix}1/\sqrt{2}\\-1/\sqrt{2}\\0\end{pmatrix},\begin{pmatrix}1/\sqrt{2}\\1/\sqrt{2}\\0\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix}\rangle$$