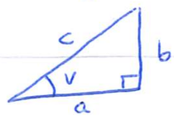
Trigonometri

Låt v vara en spetsig vinkel $(0^\circ < v < 90^\circ)$.

M.h.a. triangeln 

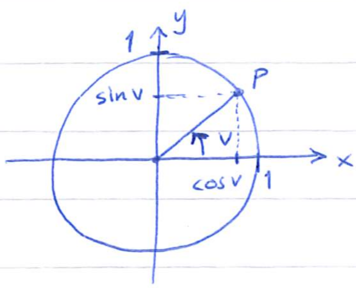
definierar vi:

$$\sin v = \frac{b}{c}, \quad \cos v = \frac{a}{c}, \quad \tan v = \frac{b}{a}, \quad \cot v = \frac{a}{b}.$$

(Obs: alla rätvinkliga trianglar med en vinkel v är likformiga (VV).)

Så
- $\sin(90^\circ - v) = \cos v$, $\cos(90^\circ - v) = \sin v$, $\tan(90^\circ - v) = \cot v$, $\cot(90^\circ - v) = \tan v$.
- $\tan v = \dfrac{\sin v}{\cos v}$, $\cot v = \dfrac{\cos v}{\sin v} = \dfrac{1}{\tan v}$.
- Trigonometriska ettan: $\sin^2 v + \cos^2 v = 1$, ty $\dfrac{a^2}{c^2} + \dfrac{b^2}{c^2} = \dfrac{a^2+b^2}{c^2} \overset{\text{Pyth.s.}}{=} \dfrac{c^2}{c^2} = 1$

Alternativ definition, i enhetscirkeln:

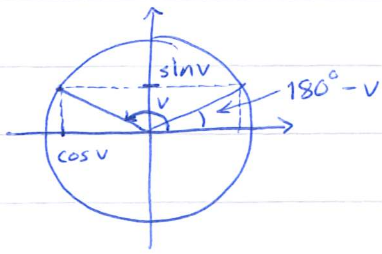


$\sin v$ = y-koordinat för P,

$\cos v$ = x-koordinat för P.

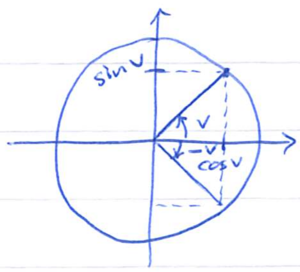
Detta utvidgar def. av $\sin v$ och $\cos v$ (och $\tan v = \dfrac{\sin v}{\cos v}$, $\cot v = \dfrac{\cos v}{\sin v}$) till alla reella tal v.

Några samband:



, så $\sin v = \sin(180^\circ - v)$,

$\cos v = -\cos(180^\circ - v)$.

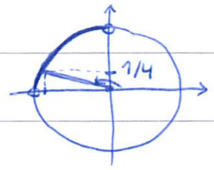


, så $\sin(-v) = -\sin v$,

$\cos(-v) = \cos v$.

Ex   Bestäm $\cos v$ om $90^\circ < v < 180^\circ$ och $\sin v = \frac{1}{4}$.

Rita upp:

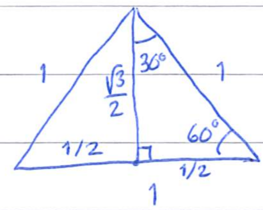


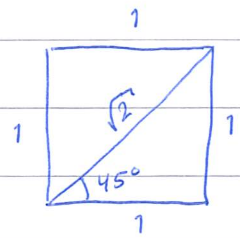
$\Rightarrow \cos v < 0$, så

$\cos v = \overset{(+)}{-}\sqrt{1-\sin^2 v} = -\sqrt{1-\frac{1}{16}} = -\frac{\sqrt{15}}{4}$.

Speciella värden:

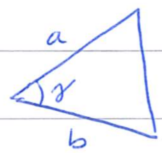
v	$\sin v$	
0°	0	$\sqrt{\frac{0}{4}}$
30°	$\frac{1}{2}$	$\sqrt{\frac{1}{4}}$
45°	$\frac{1}{\sqrt{2}}$	$\sqrt{\frac{2}{4}}$
60°	$\frac{\sqrt{3}}{2}$	$\sqrt{\frac{3}{4}}$
90°	1	$\sqrt{\frac{4}{4}}$





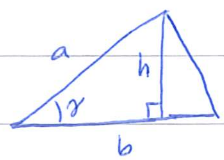
Areasatsen

Arean av triangeln



är $\frac{1}{2}ab\sin\gamma$.

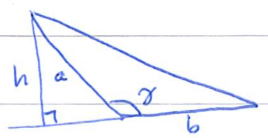
B   Om γ spetsig:



$h = a\sin\gamma$, så

arean $= \frac{1}{2}bh = \frac{1}{2}ab\sin\gamma$.

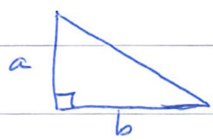
Om γ trubbig:



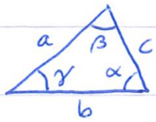
$h = a\sin(180^\circ - \gamma) = a\sin\gamma$,

så arean $= \frac{1}{2}bh = \frac{1}{2}ab\sin\gamma$.

Och $\gamma = 90^\circ$ ger arean $= \frac{1}{2}ab = \frac{1}{2}ab\sin 90^\circ$.



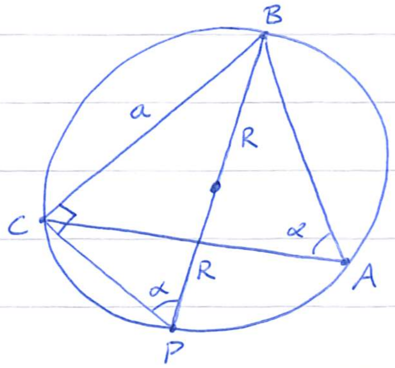
Sinussatsen

För triangeln 

gäller: $\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} = 2R$,

där R = omskrivna cirkelns radie.

B (För α spetsig) (α trubbig: övning.)



Dra diameter BP och dra PC.

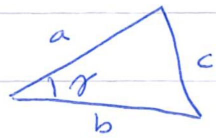
Perif.v.s. ger att $\angle CPB = \alpha$,

och att $\angle PCB = 90°$.

Så $\sin\alpha = \frac{a}{2R}$, dvs $\frac{a}{\sin\alpha} = 2R$.

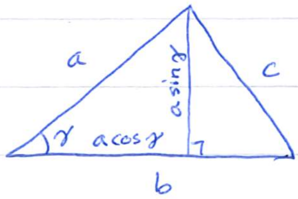
P.s.s. för $\frac{b}{\sin\beta}$ och $\frac{c}{\sin\gamma}$.

Cosinussatsen

För triangeln 

gäller: $c^2 = a^2 + b^2 - 2ab\cos\gamma$.

B (För γ spetsig och $a\cos\gamma < b$) (Tänk igenom övriga fall.)



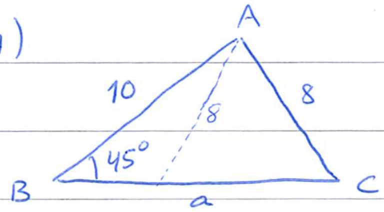
Pythagoras sats ger:

$c^2 = (a\sin\gamma)^2 + (b - a\cos\gamma)^2 =$

$= a^2\sin^2\gamma + b^2 + a^2\cos^2\gamma - 2ab\cos\gamma =$

$= a^2 + b^2 - 2ab\cos\gamma$.

Ex (6.1)



Arean av $\Delta ABC = ?$

Sätt $a = BC$. Cos.s.: $8^2 = 10^2 + a^2 - 2 \cdot 10 a \cos 45°$,

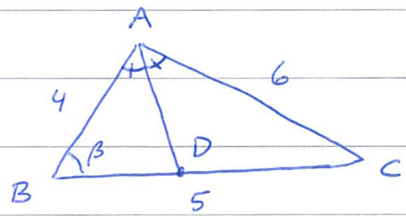
så $a^2 - 10\sqrt{2}\,a + 36 = 0$,

så $a = 5\sqrt{2} \pm \sqrt{50-36} = 5\sqrt{2} \pm \sqrt{7}\sqrt{2}$ (två möjligheter!)

Areas. ger nu:

$|\Delta ABC| = \frac{1}{2} \cdot 10 a \sin 45° = \underline{\underline{5(5 \pm \sqrt{7})}}$. (cm²)

Ex (6.2)



Längden av AD?

Bis.s.: $\frac{BD}{CD} = \frac{4}{6}$, så $BD = \frac{4}{4+6} \cdot BC = 2$.

Cos.s.: $6^2 = 4^2 + 5^2 - 2 \cdot 4 \cdot 5 \cos\beta$,

$\cos\beta = \frac{16+25-36}{2 \cdot 4 \cdot 5} = \frac{5}{2 \cdot 4 \cdot 5} = \frac{1}{8}$.

Cos.s.: $AD^2 = 4^2 + 2^2 - 2 \cdot 4 \cdot 2 \cdot \frac{1}{8} = 16 + 4 - 2 = 18$,

så $AD = \sqrt{18} = \underline{\underline{3\sqrt{2}}}$.

Formler

Sats (6.5)

$$\sin(u+v) = \sin u \cos v + \sin v \cos u$$

$$\sin(u-v) = \sin u \cos v - \sin v \cos u$$

$$\cos(u+v) = \cos u \cos v - \sin u \sin v$$

$$\cos(u-v) = \cos u \cos v + \sin u \sin v$$

$$\sin 2u = 2\sin u \cos u$$

$$\cos 2u = \cos^2 u - \sin^2 u = 2\cos^2 u - 1 = 1 - 2\sin^2 u$$

$$\tan(u+v) = \frac{\tan u + \tan v}{1 - \tan u \tan v}$$

$$\tan 2u = \frac{2\tan u}{1 - \tan^2 u}$$

B Skippar vi.

Ex Beräkna $\sin 15°$.

$$\sin 15° = \sin(45° - 30°) = \sin 45° \cos 30° - \sin 30° \cos 45° =$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} - \frac{1}{2} \cdot \frac{1}{\sqrt{2}} = \underline{\underline{\frac{\sqrt{3}-1}{2\sqrt{2}}}}.$$