**Linköpings universitet**
Magnus Berggren, MAI

# Tentamen i Matematik, fortsättningskurs. NMAA07/TEN1
## 2022-03-26, kl 8-12.

Inga hjälpmedel är tillåtna. Maclaurinutvecklingar bifogas, se baksidan.
Uppgifterna bedöms med 0 – 3 poäng.
För betyg $n$ ($n = 3, 4$ eller 5) krävs minst 4(n-1) poäng.

1) Bestäm alla primitiva funktioner till

a) $x\sin(2x)$ b) $\frac{5+x}{x^2+x-2}$ c) $\frac{\sqrt{x}}{x+4}$.

2) Beräkna

a) $\lim\limits_{x\to 0}\frac{\sin 2x-2x}{x^3}$ b) $\lim\limits_{x\to 1}\frac{e^{x-1}-\sqrt{2x-1}}{1-x+\ln x}$ c) $\int_1^{\infty} xe^{-x^2}dx$.

3) Bestäm den lösning till differentialekvationen $xy' + 2y = \frac{\ln x}{x}$, x>0, som uppfyller begynnelsevillkoret $y(1) = 3$.

4) Räkna ut volymen av den rotationskropp som uppkommer då området mellan kurvan $y = x\sqrt{x^3+1}$, $0 \le x \le 2$, och $x$-axeln roteras ett varv kring $y$-axeln.

5) Bestäm största och minsta värde av $f(x,y) = xy^2 - x + x^2 + 1$ på området $D = \{(x,y): -1 \le x \le 1,\ 0 \le y \le 2\}$.

6) Beräkna

$$\iint_D (x+y)dxdy$$

där $D$ är en triangel med hörn i punkterna (1,1), (3,1) och (1,3).

7) För vilka konstanter $a$ har funktionen $f(x) = a\ln(\cos x) + \frac{a}{2}\sin(x^2)$ lokalt maximum i origo?

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(1a) $\int x\sin 2x\,dx = \left/ \begin{array}{l} PI \\ g=x \Rightarrow g'=1 \\ f=\sin 2x \Rightarrow F=-\frac{\cos 2x}{2} \end{array} \right/ = -\frac{x}{2}\cos 2x +$

$+\int \frac{\cos 2x}{2}dx = -\frac{x}{2}\cos 2x + \frac{1}{4}\sin 2x + C.$

(1b) $\int \frac{5+x}{x^2+x-2}dx = \int \frac{5+x}{(x-1)(x+2)}dx = \int\left(\frac{2}{x-1} - \frac{1}{x+2}\right)dx =$

$= 2\ln|x-1| - \ln|x+2| + C = \ln\frac{(x-1)^2}{|x+2|} + C$

(1c) $\int \frac{\sqrt{x}}{x+4}dx = \left/ \begin{array}{l} x=t^2 \\ dx=2t\,dt \end{array} \right/ = \int \frac{t\cdot 2t}{t^2+4}dt = \int\left(2 - \frac{8}{t^2+4}\right)dt =$

$= 2t - 4\arctan\frac{t}{2} + C = 2\sqrt{x} - 4\arctan\frac{\sqrt{x}}{2} + C.$

Svar: a) $-\frac{x}{2}\cos 2x + \frac{1}{4}\sin 2x + C$

b) $\ln\frac{(x-1)^2}{|x+2|} + C$

c) $2\sqrt{x} - 4\arctan\frac{\sqrt{x}}{2} + C.$

(2a) $\lim_{x\to 0}\frac{\sin 2x - 2x}{x^3} = \lim_{x\to 0}\frac{\cancel{2x} - \frac{(2x)^3}{3!} + O(x^5) - \cancel{2x}}{x^3} = -\frac{4}{3}$

(2b) $\lim_{x\to 1}\frac{e^{x-1} - \sqrt{2x-1}}{1-x+\ln x} = \left/ \begin{array}{l} t=x-1 \\ t\to 0 \end{array} \right/ = \lim_{t\to 0}\frac{e^t - \sqrt{2t+1}}{-t+\ln(1+t)} =$

$= \lim_{t\to 0}\frac{\cancel{1}+\cancel{t}+\frac{t^2}{2}+O(t^3) - \left(\cancel{1}+\cancel{\frac{1}{2}\cdot 2t}+\frac{\frac{1}{2}(\frac{1}{2}-1)}{2}(2t)^2+O(t^3)\right)}{-\cancel{t}+\cancel{t}-\frac{t^2}{2}+O(t^3)} =$

$= \lim_{t\to 0}\frac{\frac{t^2}{2}+O(t^3)+\frac{t^2}{2}-O(t^3)}{-\frac{t^2}{2}+O(t^3)} = \lim_{t\to 0}\frac{t^2+O(t^3)}{-\frac{t^2}{2}+O(t^3)} = -2.$

(2c) $\int_1^\infty xe^{-x^2}dx = \lim_{M\to\infty}\left(-\frac{1}{2}\right)\int_1^M (-2x)e^{-x^2}dx = \left/ \begin{array}{l} t=-x^2 \\ dt=-2x\,dx \end{array} \right/ =$

$= -\frac{1}{2}\lim_{M\to\infty}\int_{-1}^{-M^2} e^t\,dt = -\frac{1}{2}\lim_{M\to\infty}\left(e^{-M^2} - e^{-1}\right) = \frac{1}{2e}.$

Svar: a) $-\frac{4}{3}$ b) $-2$ c) $\frac{1}{2e}$.

(3.) $xy' + 2y = \frac{\ln x}{x} \Leftrightarrow y' + \frac{2}{x}y = \frac{\ln x}{x^2}$ $/ \; IF = x^2 \; /$

$x^2y' + 2xy = \ln x \Leftrightarrow (x^2y)' = \ln x \Leftrightarrow$

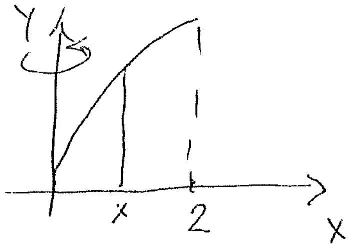
$x^2y = \int \ln x \, dx \; \Big/ \begin{matrix} PI \\ g = \ln x \Rightarrow g' = \frac{1}{x} \\ f = 1 \Rightarrow F = x \end{matrix} \Big/ = x\ln x - \int \frac{1}{x} \cdot x \, dx =$

$= x\ln x - x + C \Rightarrow y = \frac{1}{x}\ln x - \frac{1}{x} + \frac{C}{x^2}$

$y(1) = 3 \Rightarrow 3 = 1 \cdot \ln 1 - 1 + C \Rightarrow C = 4$

Svar: $y = \frac{1}{x}\ln x - \frac{1}{x} + \frac{4}{x^2}$.

(4)




$dV = 2\pi x \cdot f(x)\,dx$

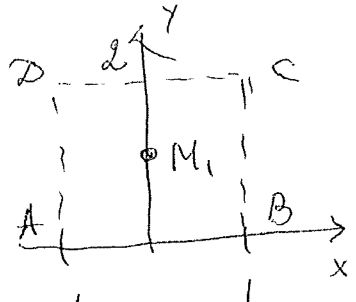
$V = 2\pi \int_0^2 x f(x)\,dx = 2\pi \int_0^2 x^2 \sqrt{x^3+1}\,dx$

$\Bigg/ \begin{matrix} t^2 = x^3 + 1 \\ 2t\,dt = 3x^2\,dx \\ x = 0 \Rightarrow \alpha = 1 \\ x = 2 \Rightarrow \beta = 3 \end{matrix} \Bigg/ = \frac{2^2}{3}\pi \int_1^3 t^2\,dt = \frac{4\pi}{9}\left(t^3\right)_1^3 = \frac{4\pi}{9} \cdot 26 =$

$= \frac{104}{9}\pi$ (v.e.)

Svar: $V = \frac{104}{9}\pi$ v.e.

(5.)




$f(x,y) = xy^2 - x + x^2 + 1, \; -1 \le x \le 1, \; 0 \le y \le 2.$

$f$ är kont. på ett kompaktområde $\Rightarrow f_{max}, f_{min}$ existerar.

$\begin{cases} f'_x = y^2 - 1 + 2x = 0 \\ f'_y = 2xy = 0 \end{cases} \Rightarrow \underset{\Uparrow}{\overset{y = \pm 1}{x = 0}}$ eller $\underset{\Uparrow}{\overset{x = \frac{1}{2}}{y = 0}} \Rightarrow$

$M_1 = (0,1) \in D$, $M_2 = (0,-1) \notin D$, $M_3 = (\frac{1}{2}, 0)$ - randpunkt

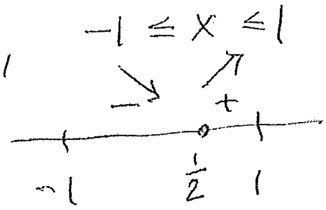
ej aktuella

$\boxed{f(0,1) = 1}$

Randpunkter:

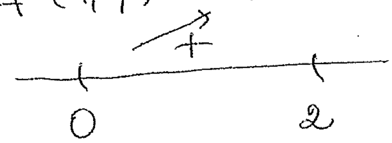
AB: $y = 0 \Rightarrow f(x,0) = x^2 - x + 1, \quad -1 \le x \le 1$

$f'(x,0) = 2x - 1 = 0 \Rightarrow x = \frac{1}{2}$




$$\boxed{\begin{array}{l} f(\frac{1}{2},0) = \frac{3}{4} \\ f(-1,0) = 3 \\ f(1,0) = 1 \end{array}}$$

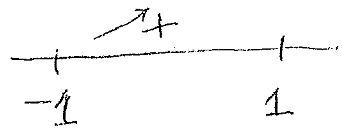
BC: $x = 1 \Rightarrow f(1,y) = y^2 + 1$, $0 \le y \le 2$ $\qquad f'(1,y) = 2y = 0 \Rightarrow y = 0$




$$\boxed{f(1,2) = 5}$$

CD: $y = 2 \Rightarrow f(x,2) = x^2 + 3x + 1, \quad -1 \le x \le 1$

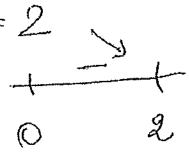
$f'(x,2) = 2x + 3 = 0 \Leftrightarrow x = -\frac{3}{2} \notin [-1,1]$




$$\boxed{f(-1,2) = -1}$$

AD: $x = -1 \Rightarrow f(-1,y) = -y^2 + 3, \quad 0 \le y \le 2$

$f'(-1,y) = -2y = 0 \Leftrightarrow y = 0 \notin ]0,2[$




Svar: $f_{max} = f(1,2) = 5, \quad f_{min} = f(-1,2) = -1.$

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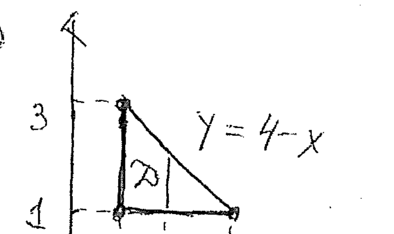


$$\iint_D (x+y)\,dx\,dy = \int_1^3 \left( \int_1^{4-x} (x+y)\,dy \right) dx$$

$$= \int_1^3 \left( xy + \frac{y^2}{2} \right)_{y=1}^{y=4-x} dx =$$

$$= \int_1^3 \left( x(3-x) + \frac{(4-x)^2}{2} - \frac{1}{2} \right) dx = \left( \frac{3}{2}x^2 - \frac{1}{3}x^3 - \frac{(4-x)^3}{6} - \frac{x}{2} \right)_1^3$$

$$= \frac{27}{2} - \frac{27}{3} - \frac{1}{6} - \frac{3}{2} - \frac{3}{2} + \frac{1}{3} + \frac{27}{6} + \frac{1}{2} = \frac{20}{3}.$$

Svar: $\frac{20}{3}$

(7.) $f(x) = a\ln(\cos x) + \frac{a}{2}\sin(x^2)$

$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} + O(x^6)$, sätt $t = -\frac{x^2}{2} + \frac{x^4}{4!} + O(x^6)$

$$\ln(1+t) = t - \frac{t^2}{2} + O(t^3) = -\frac{x^2}{2} + \frac{x^4}{4!} + O(x^6) -$$

$$- \frac{1}{2}\left(-\frac{x^2}{2} + \frac{x^4}{4!} + O(x^6)\right)^2 + O(x^6) =$$

$$= -\frac{x^2}{2} + \frac{x^4}{4!} + O(x^6) - \frac{1}{2}\left(\frac{x^4}{4} + O(x^6)\right) =$$

$$= -\frac{x^2}{2} - \frac{1}{12}x^4 + O(x^6)$$

$$f(x) = a\left(-\frac{x^2}{2} - \frac{1}{12}x^4 + O(x^6) + \frac{1}{2}(x^2 + O(x^6))\right)$$

$$= -a\cdot\frac{x^4}{12} + O(x^6) \quad \Rightarrow$$

$$f(x) - f(0) = -a\cdot\frac{x^4}{12} + O(x^6) < 0 \text{ om } a > 0$$

Svar: $a > 0$.