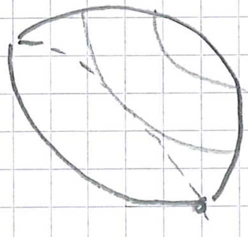
4.30 $f(x,y,z) = 3x + 2y + z \to$ max o min

då $x^2+y^2+z^2=1$ o $x+y+z \geq 1$.

$$\underbrace{(\text{en sfär}) \qquad (\text{ett halvrum})}_{D}$$

D: 

är kompakt
f är kontinuerlig $\Rightarrow$ f har max o min.

Extrempunkter ligger bland lösningar till följande system:

I) $\begin{cases} \nabla f, \nabla g, \nabla h \text{ är lin. beroende} \\ g=1,\ h=1, \text{ där } g = x^2+y^2+z^2,\ h = x+y+z \end{cases}$

II) $\begin{cases} \nabla f, \nabla g \text{ är lin beroende} \\ g=1,\ h>1 \end{cases}$

Lös (I): $\begin{cases} \begin{vmatrix} f'_x & f'_y & f'_z \\ g'_x & g'_y & g'_z \\ h'_x & h'_y & h'_z \end{vmatrix} = 0 \\ g=1,\ h=1 \end{cases} \Leftrightarrow \begin{cases} \begin{vmatrix} 3 & 2 & 1 \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{vmatrix} = 0 \\ g=1,\ h=1 \end{cases}$

$\Leftrightarrow \begin{cases} x - 2y + z = 0 & (1) \\ x^2+y^2+z^2=1 & (2) \\ x+y+z=1 & (3) \end{cases}$ $\quad \left.\begin{array}{l} (1) \text{ o } (3) \Rightarrow y = \frac{1}{3} \\ (1) \Rightarrow x = \frac{2}{3} - z \end{array}\right] \to (2) \Rightarrow$

$\left(\frac{2}{3} - z\right)^2 + \frac{1}{9} + z^2 = 1 \Leftrightarrow z_{1,2} = \frac{1 \pm \sqrt{3}}{3}, \Rightarrow$

$$P_1\left(\frac{1-\sqrt{3}}{3}, \frac{1}{3}, \frac{1+\sqrt{3}}{3}\right),\ P_2\left(\frac{1+\sqrt{3}}{3}, \frac{1}{3}, \frac{1-\sqrt{3}}{3}\right)$$

Räkna f's värde: $f(P_1) = 3\left(\frac{1-\sqrt{3}}{3}\right) + \frac{2}{3} + \frac{1+\sqrt{3}}{3} =$

$= \underline{2 - \frac{2}{3}\sqrt{3}}$, $f(P_2) = 3\,\frac{(1+\sqrt{3})}{3} + \frac{2}{3} + \frac{1-\sqrt{3}}{3} = \underline{2 + \frac{2}{3}\sqrt{3}}$

(Kandidater).

Lös (II): $\begin{cases} k \cdot \nabla f = \nabla g \\ g = 1,\ h > 1 \end{cases}$, där k är Lagranges multiplikator.

$$\Leftrightarrow \begin{cases} 3k = 2x \\ 2k = 2y \\ k = 2z \\ x^2+y^2+z^2 = 1 \quad (1) \\ x+y+z > 1 \quad (2) \end{cases} \Rightarrow \begin{array}{l} x = \frac{3}{2}k \\ y = k \\ z = \frac{1}{2}k \end{array} \xrightarrow{(1)} \Rightarrow \left(\frac{3}{2}k\right)^2 + k^2 + \left(\frac{k}{2}\right)^2 = 1$$

$\Rightarrow k_{1,2} = \pm\sqrt{\frac{2}{7}} \Rightarrow P_3\left(\frac{3}{2}\sqrt{\frac{2}{7}}, \sqrt{\frac{2}{7}}, \frac{1}{2}\sqrt{\frac{2}{7}}\right)$ kandidat

$P_4\left(-\frac{3}{2}\sqrt{\frac{2}{7}}, -\sqrt{\frac{2}{7}}, -\frac{1}{2}\sqrt{\frac{2}{7}}\right)$ ingen kandidat

$\text{ty } \underline{x+y+z < 0}$ (Ols (2))

Räkna $f(P_3) = \left(\frac{9}{2} + 2 + \frac{1}{2}\right)\sqrt{\frac{2}{7}} = 7\cdot\sqrt{\frac{2}{7}} = \underline{\sqrt{14}}$

(kandidatvärde)

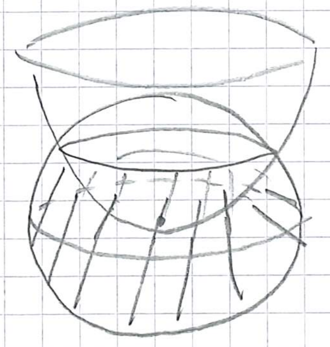
Välj max o min av

$\left\{2 - \frac{2}{3}\sqrt{3},\ 2 + \frac{2}{3}\sqrt{3},\ \sqrt{14}\right\} \Rightarrow \max_D f = \sqrt{14}$

$\min_D f = 2 - \frac{2}{3}\sqrt{3}$

4.32 $f(x,y,z) = x+y+z \longrightarrow$ max o min

då $\underbrace{\underset{\text{(ett klot)}}{x^2+y^2+z^2 \le 2} \quad \text{o} \quad \underset{\text{(under paraboloiden)}}{x^2+y^2 \le z}}_{\mathcal{D}}$

$\mathcal{D} =$  skål (ett kompakt)

$$\left.\begin{array}{l} \mathcal{D} = \text{skål (ett kompakt)} \\ f \text{ är kontinuerlig} \end{array}\right] \Rightarrow f \text{ har max o min}$$

Extrempunkter ligger bland lösningar till följande system:

$$(\mathrm{I}) \quad \begin{cases} \nabla f, \nabla g, \nabla h \text{ är lin. beroende} \\ g = 2, \quad h = 0, \quad \text{där } g = x^2+y^2+z^2 \\ \qquad\qquad\qquad\qquad\quad h = x^2+y^2-z \end{cases}$$

$$(\mathrm{II}) \quad \begin{cases} \nabla f, \nabla g \text{ är lin. beroende} \\ g = 2, \quad h < 0 \end{cases}$$

$$(\mathrm{III}) \quad \begin{cases} \nabla f, \nabla h \text{ är lin. beroende} \\ g < 2, \quad h = 0 \end{cases}$$

Lös (I):

$$\begin{cases} \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ 2x & 2y & -1 \end{vmatrix} = 0 \\ g = 2, h = 0 \end{cases} \Leftrightarrow \begin{cases} (x-y) + 2z(x-y) = 0 \\ g = 2, h = 0 \end{cases}$$

$$\Updownarrow$$

$$\begin{cases} (x-y)(1+2z) = 0 & (1) \\ x^2+y^2+z^2 = 2 & (2) \\ x^2+y^2-z = 0 & (3) \end{cases}$$

$(2)-(3)$: $z^2+z-2=0 \iff z_{1,2}=\underbrace{-2,\ 1}_{\uparrow}$

$(1) \Rightarrow x=y \to (3)$ tillsammans med

$\Rightarrow$
- $2x^2=1 \Rightarrow x_{1,2}=\pm\frac{1}{\sqrt{2}}$
- $2x^2=-2 \Rightarrow$ inga lösningar

$\Rightarrow$

$P_1\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1\right)$, $P_2\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 1\right)$

Räkna f's värde: $f(P_1)=\underline{\sqrt{2}+1}$, $f(P_2)=\underline{-\sqrt{2}+1}$

(kandidater)

(II)
$$\begin{cases} k\cdot 1=2x \\ k\cdot 1=2y \\ k\cdot 1=2z \\ \underbrace{x^2+y^2+z^2=2}_{(1)},\quad \underbrace{x^2+y^2-z<0}_{(2)} \end{cases} \Rightarrow \begin{array}{l} x=y=z=\frac{k}{2} \to (1) \Rightarrow \\ 3\,\frac{k^2}{4}=2, \text{ eller } k_{1,2}=\pm\sqrt{\frac{8}{3}} \end{array}$$

$P_3\left(\sqrt{\frac{8}{3}}, \sqrt{\frac{8}{3}}, \sqrt{\frac{8}{3}}\right)$, $P_4\left(-\sqrt{\frac{8}{3}}, -\sqrt{\frac{8}{3}}, -\sqrt{\frac{8}{3}}\right)$

<u>Obs</u> (2): $\frac{8}{3}+\frac{8}{3} > \pm\sqrt{\frac{8}{3}}$ (inga kandidater)

(III)
$$\begin{cases} k\cdot 1=2x \\ k\cdot 1=2y \\ k\cdot 1=-1 \\ \underbrace{x^2+y^2+z^2<2}_{(1)},\quad \underbrace{x^2+y^2-z=0}_{(2)} \end{cases} \Rightarrow x=y=\frac{k}{2}=-\frac{1}{2} \to (2) \Rightarrow z=\frac{1}{2}$$

$\Downarrow$

$P_5\left(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right)$

Kontrollera (1): $\frac{1}{4}+\frac{1}{4}+\frac{1}{4}<2$ (ok)

Räkna f's värde: $f(P_5)=-\frac{1}{2}-\frac{1}{2}+\frac{1}{2}=-\frac{1}{2}$

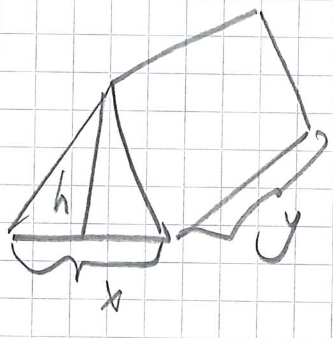
(kandidat värde)

Välj max o min av

$\{\sqrt{2}+1, -\sqrt{2}+1, -\frac{1}{2}\} \Rightarrow \max_{D} f = 1+\sqrt{2}$

$\min_{D} f = -\frac{1}{2}$

4.34.



$V = \frac{xh}{2} \cdot y$ (fixerad)

$A = 2 \cdot \frac{xh}{2} + 2y\sqrt{\frac{x^2}{4}+h^2}$

$= xh + 2y\sqrt{\frac{x^2}{4}+h^2}$

Finn h s.a. A är minst.

$y = \frac{2V}{xh} \to A \Rightarrow A(x,h) = xh + \frac{4V}{xh} \cdot \sqrt{\frac{x^2}{4}+h^2} =$

$= xh + 4V \cdot \sqrt{\frac{1}{4h^2} + \frac{1}{x^2}}$

$\nabla A = \bar{0} \Leftrightarrow \begin{cases} A'_x = 0 \\ A'_y = 0 \end{cases} \Leftrightarrow \begin{cases} h + 4V \cdot \frac{1}{2}(\ldots)^{-1/2} \cdot \left(-\frac{2}{x^3}\right) = 0 \\ x + 4V \cdot \frac{1}{2}(\ldots)^{-1/2} \cdot \left(-\frac{1}{2h^3}\right) = 0 \end{cases}$

$\Leftrightarrow \begin{cases} hx^3 - 4V \cdot (\ldots)^{-1/2} = 0 \quad (1) \\ xh^3 - V(\ldots)^{-1/2} = 0 \end{cases}$ $hx^3 = 4xh^3$ eller $x^2 = 2h^2$

$\boxed{x = 2h}$ Obs $x = -2h$ passar ej ty $x, h > 0$

$\to (1): h(2h)^3 - 4V\left(\frac{1}{4h^2} + \frac{1}{4h^2}\right)^{-1/2} = 0$

$8h^4 - 4V\left(\frac{1}{2h^2}\right)^{-1/2} = 0$ eller $8h^4 - 4V \cdot h\sqrt{2} = 0 \Rightarrow$

$h = \left(\frac{V}{\sqrt{2}}\right)^{1/3}$