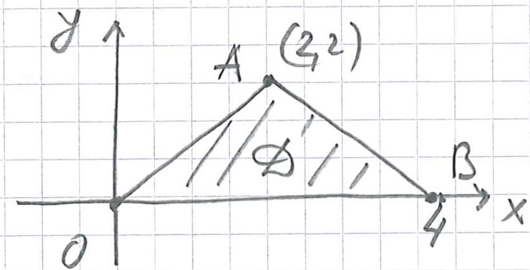
5.33 $I = \iint_D e^{(y-x)/(y+x)} \, dx\,dy$, $D$ är triangeln med hörn i $(0,0)$, $(4,0)$, $(2,2)$

Rita!




Beskriv sidorna OA, AB, OB:

OA ligger på linjen: $y = x$ eller $y - x = 0$

AB — " — : $x + y = 4$ Obs V.L.

OB — " : $y = 0$

Inför variabelbytet: $u = y - x$, $v = x + y$

$\Leftrightarrow \begin{cases} \frac{v-u}{2} = x \\ \frac{u+v}{2} = y \end{cases}$ $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix}$

$\frac{\partial(x,y)}{\partial(u,v)} = -\frac{1}{2}$, $\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{2}$

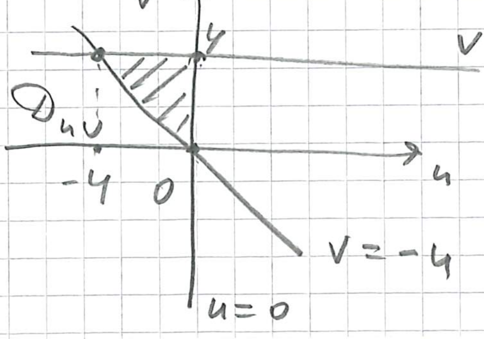
belopp!

$\Rightarrow I = \iint_{D_{uv}} e^{\frac{u}{v}} \cdot \frac{1}{2} \, du\,dv$

$D_{uv} = ?$

$D_{uv}$ avgränses av linjerna: $u = 0$, $v = 4$,

$\frac{u+v}{2} = 0$ (eller $u + v = 0$)




$\Rightarrow D_{uv} = \begin{cases} -4 \leq u \leq 0 \\ -u \leq v \leq 4 \end{cases}$ eller

$\begin{cases} -v \leq u \leq 0 \\ 0 \leq v \leq 4 \end{cases}$

$$I = \frac{1}{2}\int_0^4 \left( \int_{-v}^{0} e^{u/v}\,du \right) dv = \left| \begin{array}{ll} t = \frac{u}{v}, & du = v\,dt \\ dt = \frac{1}{v}\,du & \end{array} \right| =$$

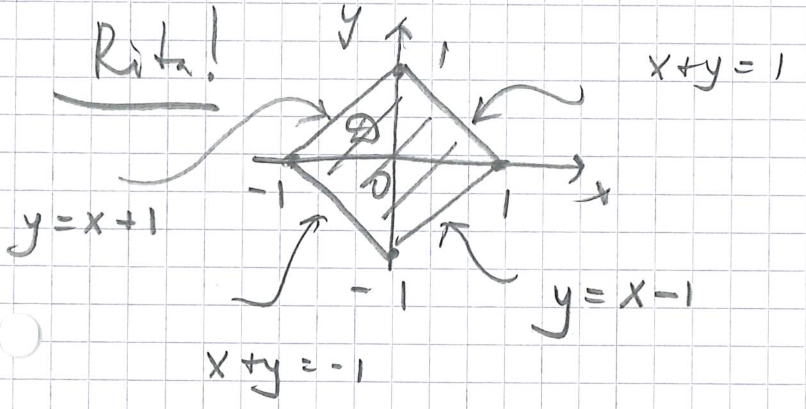
$$= \frac{1}{2}\int_0^4 \left( \int_{u=-v}^{u=0} e^{t} \cdot v\,dt \right) dv = \frac{1}{2}\int_0^4 v \cdot e^{t} \Big|_{u=-v}^{u=0} dv =$$

$$= \frac{1}{2}\int_0^4 v \cdot e^{u/v} \Big|_{-v}^{0} dv = \frac{1}{2}\int_0^4 v(1-e^{-1})\,dv =$$

$$= \frac{1-e^{-1}}{2} \cdot \frac{v^2}{2}\Big|_0^4 = 4(1-e^{-1})$$

<u>5.34.</u> $I = \iint_D e^{x-y}\,dx\,dy, \quad D = \{ |x| + |y| \le 1 \}$

<u>Rita!</u>




Betrakta variabelbytet:

$$\begin{cases} x - y = u \\ x + y = v \end{cases} \Leftrightarrow \begin{cases} x = \frac{u+v}{2} \\ y = \frac{v-u}{2} \end{cases}$$

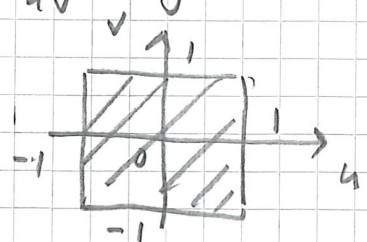
$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix}, \quad \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{2} = \left| \frac{\partial(x,y)}{\partial(u,v)} \right|$$

belopp

$$\Rightarrow I = \iint_{D_{uv}} e^{u} \cdot \frac{1}{2}\,du\,dv$$

$D_{uv} = ?$

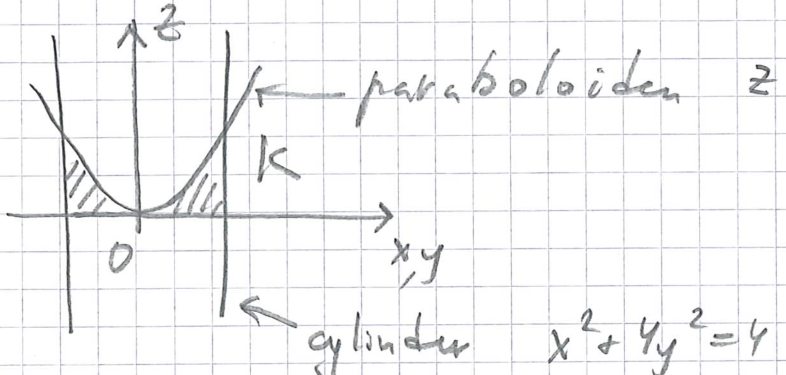
$D_{uv}$ begränsas av linjerna: $v = 1$, $v = -1$, $u = 1$, $u = -1$.



$$\Rightarrow I = \int_{-1}^{1}\left(\int_{-1}^{1} e^{u}\cdot\frac{1}{2}\,dv\right)du = \frac{1}{2}\int_{-1}^{1} e^{u}\cdot v\Big]_{-1}^{1}\,du =$$

$$= \int_{-1}^{1} e^{u}\,du = e^{u}\Big|_{-1}^{1} = e - e^{-1}$$

5.35 (a) (5.16 (c) med variabelbyte)



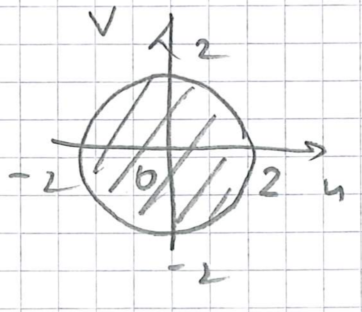

Volymen av K $= V = \iint_{D} (x^2 + 4y^2)\,dx\,dy$, där

$$D = \{x^2 + 4y^2 \le 4\}.$$

1:a substitutionen: $\begin{cases} u = x \\ v = 2y \end{cases} \Leftrightarrow \begin{cases} x = u \\ y = v/2 \end{cases}$

$$\frac{\partial xy}{\partial uv} = \begin{vmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{vmatrix}, \qquad \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{2}$$

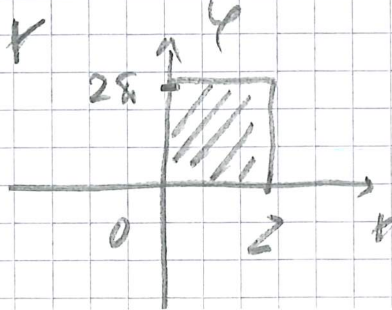
$$D_{uv} = \{u^2 + v^2 \le 2^2\} \Rightarrow V = \iint_{D_{uv}} (u^2 + v^2)\cdot\frac{1}{2}\,du\,dv.$$




2:a substitutionen: $\begin{cases} u = r\cos\varphi \\ v = r\sin\varphi \end{cases}$

$$\frac{\partial(u,v)}{\partial(r,\varphi)} = r$$

$D_{r\varphi} =$

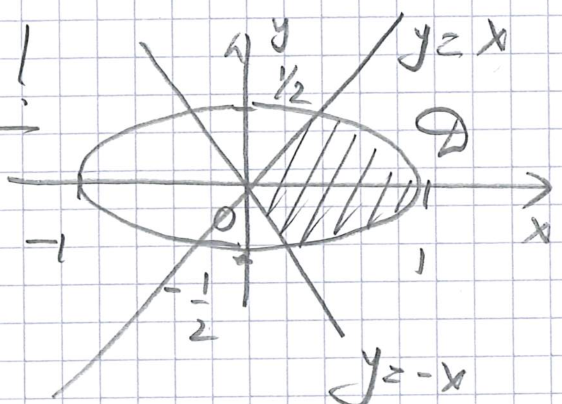


$$V = \frac{1}{2}\int_0^2 \left(\int_0^{2\pi} r^2 \cdot r\,d\varphi\right) dr = \frac{1}{2}\int_0^2 r^3\varphi\Big|_0^{2\pi} dr =$$

$$= \pi \cdot \frac{r^4}{4}\Big|_0^2 = 4\pi.$$

**5.37** $\quad I = \iint\limits_{D} x^2\,dx\,dy, \quad D = \{x^2+4y^2\le 1,\ |y|\le x\}.$

Rita!




$x^2+4y^2=1$ (en ellip)

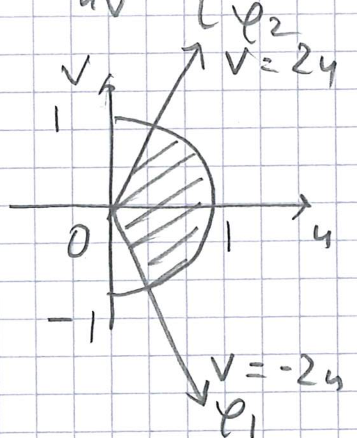
eller

$$x^2 + \frac{y^2}{(\frac{1}{2})^2} = 1$$

$|y|\le x \iff -x\le y\le x, \quad x\ge 0$

1:a substitutionen: $\begin{cases} u = x \\ v = 2y \end{cases} \iff \begin{cases} x = u \\ y = \frac{v}{2} \end{cases}$

$D_{uv} = \{u^2+v^2\le 1,\ |v|\le 2u\}, \quad \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{2}$




$$I = \iint\limits_{D_{uv}} u^2\cdot\frac{1}{2}\,du\,dv.$$

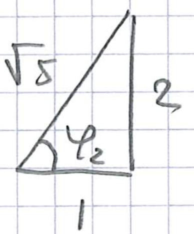
2:a substitutionen: $\begin{cases} u = r\cos\varphi \\ v = r\sin\varphi \end{cases}$

$$\Rightarrow I = \frac{1}{2}\int_0^1\left(\int_{\varphi_1}^{\varphi_2} r^2\cos^2\varphi\ \underset{\text{Obs!}}{\underline{\underline{r}}}\,d\varphi\right)dr =$$

$$\frac{1}{2}\int_0^1 r^3\left(\int_{\varphi_1}^{\varphi_2}\cos^2\varphi\,d\varphi\right)dr = \frac{1}{2}\int_{\varphi_1}^{\varphi_2}\cos^2\varphi\,d\varphi\cdot\int_0^1 r^3\,dr =$$

$$= \frac{1}{4}\int_{\varphi_1}^{\varphi_2}(1+\cos 2\varphi)\,d\varphi\cdot\frac{1}{4} = \frac{1}{16}\left(\varphi+\frac{\sin 2\varphi}{2}\right)\Big|_{\varphi_1}^{\varphi_2} =$$

$$= \frac{1}{16}\left((\varphi_2-\varphi_1)+\sin 2\varphi_2\right)$$

1) 


$$\Rightarrow \quad \sin\varphi_2 = \frac{2}{\sqrt{5}}, \quad \cos\varphi_2 = \frac{1}{\sqrt{5}} \quad \Rightarrow \quad \sin 2\varphi_2 = 2\cdot\frac{2}{\sqrt{5}}\cdot\frac{1}{\sqrt{5}} = \frac{4}{5}$$

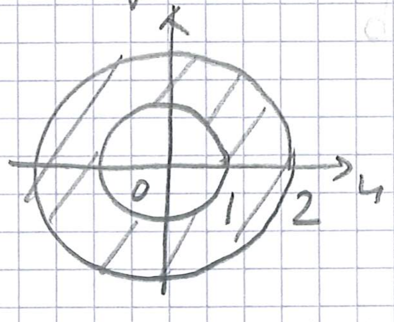
2) $\varphi_2 - \varphi_1 = 2\varphi_2 = 2\arctan 2$

$$\Rightarrow \quad \underline{I} = \frac{1}{16}\left(2\arctan 2 + 4/5\right) = \frac{1}{8}\arctan 2 + \frac{1}{20}$$

<u>5.38</u> $\quad \underline{I} = \iint_{\mathcal{D}} \frac{dx\,dy}{\sqrt{(x+2y)^2+4y^2+1}}, \quad \mathcal{D} = \{1 \le (x+2y)^2 + 4y^2 \le 4\}.$

1:a <u>Substitutionen</u>: $\begin{cases} u = x+2y \\ v = 2y \end{cases} \Leftrightarrow \begin{cases} x = u-v \\ y = \frac{v}{2} \end{cases}$

$$\mathcal{D}_{uv} = \{1 \le u^2+v^2 \le 4\}$$




$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{2}$$

$$\underline{I} = \iint_{\mathcal{D}_{uv}} \frac{1}{\sqrt{u^2+v^2+1}}\cdot\frac{1}{2}\,du\,dv = \Big/ \text{ 2:a substitutionen } u = r\cos\varphi,\ v = r\sin\varphi \Big/ =$$

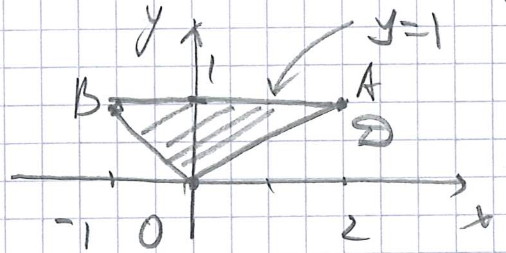
$$= \frac{1}{2}\int_1^2\left(\int_0^{2\pi}\frac{1}{\sqrt{r^2+1}}\cdot r\,d\varphi\right)dr = \frac{1}{2}\int_1^2\frac{r}{\sqrt{r^2+1}}\cdot 2\pi\,dr =$$

$$= \left| \begin{array}{ll} t = r^2+1 & r\,dr = \dfrac{dt}{2} \\ dt = 2r\,dr & \end{array} \right| = \pi \int_{r=1}^{r=2} \frac{1}{\sqrt{t}} \cdot \frac{dt}{2} =$$

$$= \frac{\pi}{2} \left. \frac{t^{1/2}}{1/2} \right|_{r=1}^{r=2} = \pi \cdot \left(r^2+1\right) \Big|_1^2 = \pi\left(\sqrt{5} - \sqrt{2}\right)$$

5.40 (b) $\quad I = \iint_D \dfrac{1}{1+(x-2y)^2}\,dx\,dy$, $D$ är triangeln med hörn i $(0,0)$, $(2,1)$ o $(-1,1)$

Rita!




Obs 1) $OA$ ligger på linjen $y = \frac{x}{2}$ ①

$OB$ —"— $y = -x$,

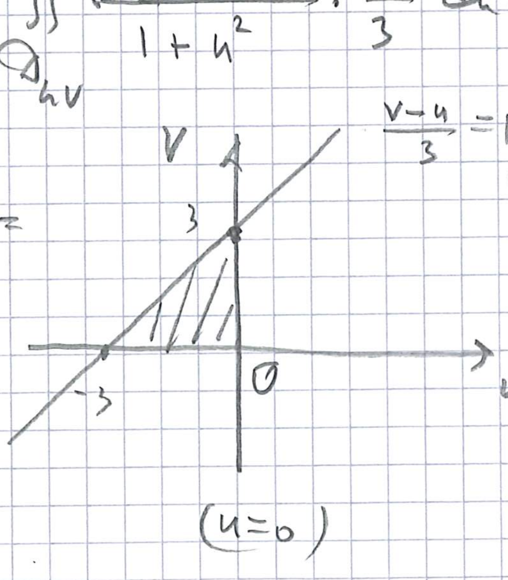
2) $y = \frac{x}{2} \iff x - 2y = 0$

$y = -x \iff x + y = 0$

Använd substitution:
$$\begin{cases} u = x - 2y \\ v = x + y \end{cases} \iff \begin{array}{l} x = \dfrac{u+2v}{3} \\ y = \dfrac{v-u}{3} \end{array}$$

$$I = \iint_{D_{uv}} \frac{1}{1+u^2} \cdot \frac{1}{3}\,du\,dv, \qquad \frac{\partial xy}{\partial uv} = \begin{vmatrix} \frac{1}{3} & \frac{2}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{vmatrix} = \underline{\underline{\frac{1}{3}}}$$

$D_{uv} =$




$\frac{v-u}{3} = 1$ eller $\underline{v = u+3}$

$\Rightarrow$

$$I = \frac{1}{3}\int_{-3}^{0}\left(\int_{0}^{u+3}\frac{1}{1+u^2}\,dv\right)du = \frac{1}{3}\int_{-3}^{0}\frac{1}{1+u^2}\cdot v\Big|_{0}^{u+3}\,du =$$

$$= \frac{1}{3}\int_{-3}^{0}\frac{u+3}{1+u^2}\,du = \frac{1}{3}\cdot\frac{1}{2}\ln(1+u^2)\Big|_{-3}^{0} + \frac{1}{3}\cdot 3\arctan u\Big|_{-3}^{0}$$

$$= -\frac{1}{6}\ln 10 + \arctan 3$$