5.43

$$I = \iiint_{\Omega} x \, dx\,dy\,dz, \quad \Omega = \left\{ \begin{array}{l} 0 \le x \le 1 \\ 0 \le y \le 1-x \\ 2y \le z \le 1+y^2 \end{array} \right\}$$

Skriv om integralen i upprepad form:

$$I = \underbrace{\int_0^1 \Big(\underbrace{\int_0^{1-x} \Big(\underbrace{\int_{2y}^{1+y^2} x \, dz}_{\text{inre}} \Big) dy}_{\text{mellan}} \Big) dx}_{\text{yttre}} =$$

$$= \int_0^1 x \Big(\int_0^{1-x} z \Big|_{2y}^{1+y^2} dy \Big) dx = \int_0^1 x \Big(\int_0^{1-x} \underbrace{(1+y^2-2y)}_{(y-1)^2} dy \Big) dx$$

$$= \int_0^1 x \cdot \frac{(y-1)^3}{3} \Big|_0^{1-x} dx = \frac{1}{3} \int_0^1 x \cdot (-x^3+1)\, dx = \frac{1}{3} \int_0^1 (x - x^4)\, dx =$$

$$= \frac{1}{3} \Big(\frac{x^2}{2} - \frac{x^5}{5} \Big) \Big|_0^1 = \frac{1}{3} \Big(\frac{1}{2} - \frac{1}{5} \Big) = \frac{1}{10}$$

5.44

$$I = \iiint_{\Omega} x^2 y \, e^{xyz} \, dx\,dy\,dz, \quad \Omega = \{ 0 \le x, y, z \le 1 \}$$

(en kub)

Upprepade formen:

$$I = \int_0^1 \Big(\int_0^1 \Big(\int_0^1 x^2 y \, e^{xyz} \, dz \Big) dy \Big) dx = \left| \begin{array}{l} t = xyz \\ dt = xy\, dz \end{array} \right|$$

$$= \int_0^1 x \Big(\int_0^1 \Big(\int_{z=0}^{z=1} e^t \, dt \Big) dy \Big) dx = \int_0^1 x \Big(\int_0^1 e^{xyz} \Big|_0^1 dy \Big) dx =$$

$$= \int_0^1 x\left(\int_0^1 (e^{xy}-1)\,dy\right)dx = \int_0^1\left(\int_0^1 x e^{xy}\,dy\right)dx -$$

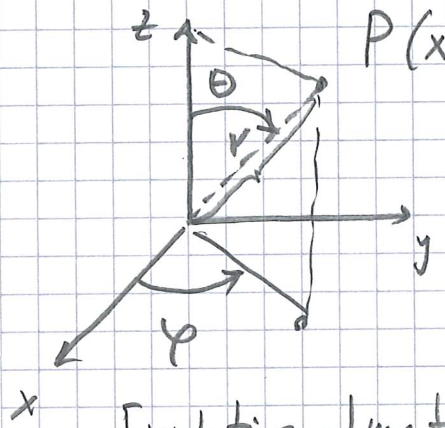
$$t = xy$$

$$- \int_0^1 x\left(\underbrace{\int_0^1 dy}_{=1}\right)dx = \int_0^1\left(\int_{y=0}^{y=1} e^t\,dt\right)dx - \frac{x^2}{2}\Big|_0^1 =$$

$$= \int_0^1 e^{xy}\Big|_0^1\,dx - \frac{1}{2} = \int_0^1 (e^x-1)\,dx - \frac{1}{2} =$$

$$= (e^x - x)\Big|_0^1 - \frac{1}{2} = e^1 - 1 - 1 - \frac{1}{2} = e - \frac{5}{2}$$

5.45 Sfäriska koordinater:



$P(x,y,z)$ $\quad x = r\cos\varphi\sin\theta$

$y = r\sin\varphi\sin\theta \qquad (r,\varphi,\theta)$

$z = r\cos\theta$

Funktionalmatris:

$$\begin{bmatrix} x'_r & x'_\varphi & x'_\theta \\ y'_r & y'_\varphi & y'_\theta \\ z'_r & z'_\varphi & z'_\theta \end{bmatrix} =$$

$$= \begin{bmatrix} \cos\varphi\sin\theta & -r\sin\varphi\sin\theta & r\cos\varphi\cos\theta \\ \sin\varphi\sin\theta & r\cos\varphi\sin\theta & r\sin\varphi\cos\theta \\ \cos\theta & 0 & -r\sin\theta \end{bmatrix} = A$$

(inför)

Obs <u>Funktional determinant</u> = det(funkt. matris)

$\det A = \Big/$ utveckla efter 3:raden $\Big/ =$

$$\cos\theta \cdot \begin{vmatrix} -r\sin\varphi\sin\theta & r\cos\varphi\cos\theta \\ r\cos\varphi\sin\theta & r\sin\varphi\cos\theta \end{vmatrix} +$$

$$(-r\sin\theta) \cdot \begin{vmatrix} \cos\varphi\sin\theta & -r\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & r\cos\varphi\sin\theta \end{vmatrix} =$$

$$= \cos^2\theta \cdot \sin\theta\left(-r^2\sin^2\varphi - r^2\cos^2\varphi\right)$$

$$- r^2\sin^3\theta\,(\underbrace{\cos^2\varphi + \sin^2\varphi}_{1}) = -r^2\sin\theta\,(\underbrace{\cos^2\theta + \sin^2\theta}_{1})$$

$$= -r^2\sin\theta$$

Vid variabelbyte använder man

$$\left|\det(\text{funkt. matris})\right| = \left|-r^2\sin\theta\right| = r^2\cdot\sin\theta$$

belopp

ty $0 \le \theta \le \pi$

<u>5.46</u> $I = \iiint_\Omega xyz\,dx\,dy\,dz$, Ω = den del av <u>enhetsklotet</u> som ligger i $\underbrace{\text{1:a oktanten}}_{x,y,z \ge 0.}$

$x^2+y^2+z^2 \le 1$

Använd sfäriska koordinater:

$$I = \iiint_{\Omega_{r\varphi\theta}} \underbrace{r\cos\varphi\sin\theta}_{x} \cdot \underbrace{r\sin\varphi\sin\theta}_{y} \cdot \underbrace{r\cos\theta}_{z} \cdot \underbrace{r^2\sin\theta}_{\det(f.m)} \cdot dr\,d\varphi\,d\theta$$

$$= \left| \underline{\text{Obs}} \quad \Omega_{r\varphi\theta} = \begin{cases} 0 \le r \le 1 \\ 0 \le \varphi \le \frac{\pi}{2} \\ 0 \le \theta \le \frac{\pi}{2} \end{cases} \right| =$$

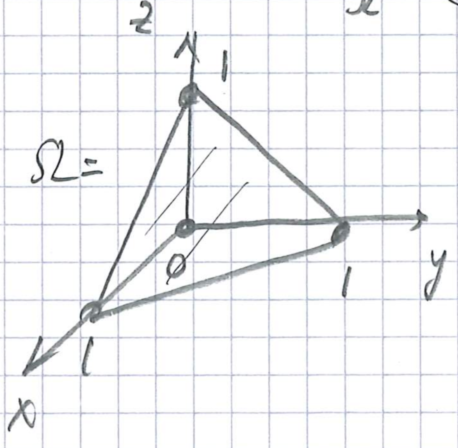
$$= \int_0^1 \left(\int_0^{\pi/2} \left(\int_0^{\pi/2} r^5 \cos\varphi \cdot \sin\varphi \cdot \sin^3\theta \cdot \cos\theta \, d\theta \right) d\varphi \right) dr$$

$$= \int_0^1 r^5\,dr \cdot \int_0^{\pi/2} \cos\varphi \sin\varphi \, d\varphi \cdot \int_0^{\pi/2} \sin^3\theta \cos\theta \, d\theta$$

$t = \sin\varphi$

$$= \frac{1}{6} \cdot \underbrace{\frac{\sin^2\varphi}{2}\Big|_0^{\pi/2}}_{= \frac{1}{2}} \cdot \underbrace{\frac{\sin^4\theta}{4}\Big|_0^{\pi/2}}_{= \frac{1}{4}} = \frac{1}{6} \cdot \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{48}$$

<u>5.47</u> $I = \iiint_\Omega \frac{1}{(1+x+y+z)^3}\,dx\,dy\,dz$, Ω är <u>tetraedern</u> med hörn i $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$

$\Omega =$ 

Obs $\Omega = \left\{ \begin{array}{l} 0 \le x \le 1 \\ 0 \le y \le 1-x \\ 0 \le z \le 1-x-y \end{array} \right\} \Rightarrow$

Upprepade formen: $I = \int_0^1 \left(\int_0^{1-x} \left(\int_0^{1-x-y} \frac{1}{(1+x+y+z)^3} dz \right) dy \right) dx$

$$= \left/ \begin{array}{l} t = 1+x+y+z \\ dt = dz \end{array} \right/ = \int_0^1 \left(\int_0^{1-x} \left(\int_{z=0}^{z=1-x-y} t^{-3} dt \right) dy \right) dx =$$

$$= \int_0^1 \left(\int_0^{1-x} \frac{t^{-2}}{-2} \Big/_{z=0}^{z=1-x-y} dy \right) dx =$$

$$-\frac{1}{2} \int_0^1 \left(\int_0^{1-x} \frac{1}{(1+x+y+z)^2} \Big/_0^{1-x-y} dy \right) dx =$$

$$= -\frac{1}{2} \int_0^1 \left(\int_0^{1-x} \left(\frac{1}{2^2} - \frac{1}{(1+x+y)^2} \right) dy \right) dx =$$

$$= -\frac{1}{8} \int_0^1 \left(\int_0^{1-x} dy \right) dx + \frac{1}{2} \int_0^1 \left(\int_0^{1-x} \frac{1}{(1+x+y)^2} dy \right) dx = \quad /t = 1+x+y/$$

$$= -\frac{1}{16} + \frac{1}{2} \int_0^1 \left(\int_{y=0}^{y=1-x} t^{-2} dt \right) dx =$$

$$= -\frac{1}{16} + \frac{1}{2} \int_0^1 \left(-\frac{1}{t} \right) \Big/_{y=0}^{y=1-x} dx = -\frac{1}{16} - \frac{1}{2} \int_0^1 \frac{1}{1+x+y} \Big/_0^{1-x} dx =$$

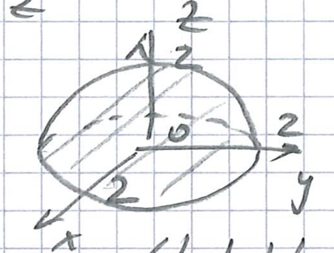
$$= -\frac{1}{16} - \frac{1}{2}\int_0^1 \left(\frac{1}{2} - \frac{1}{1+x}\right) dx = -\frac{1}{16} - \frac{1}{4} + \frac{1}{2}\ln(1+x)\Big|_0^1 =$$

$$= \frac{1}{2}\ln 2 - \frac{5}{16}$$

<u>5.48</u>

$$I = \iiint_\Omega (x^2+y^2-z^2)\,dx\,dy\,dz$$

$$\Omega = \{\, x^2+y^2+z^2 \le 4,\ z \ge 0 \,\}$$



Använd sfäriska koordinater: (halvklot)

$$\Omega_{r\varphi\theta} = \left\{ \begin{array}{l} 0 \le r \le 2 \\ 0 \le \varphi \le 2\pi \\ 0 \le \theta \le \pi/2 \end{array} \right\}$$

$$\Rightarrow I = \int_0^2 \left(\int_0^{2\pi} \left(\int_0^{\pi/2} \Big(\underbrace{(r\cos\varphi\sin\theta)^2}_{x^2} + \underbrace{(r\sin\varphi\sin\theta)^2}_{y^2} - \underbrace{(r\cos\theta)^2}_{z^2} \Big) r^2 \sin\theta \, d\theta \right) d\varphi \right) dr =$$

$$\int_0^2 \left(\int_0^{2\pi} \left(\int_0^{\pi/2} (r^2\sin^2\theta - r^2\cos^2\theta)\, r^2 \sin\theta \, d\theta \right) d\varphi \right) dr =$$

$$= \int_0^2 r^4\,dr \cdot \int_0^{2\pi} 1\,d\varphi \cdot \int_0^{\pi/2} (1 - 2\cos^2\theta)\sin\theta\,d\theta = \quad / t = \cos\theta /$$

$$= \frac{2^5}{5} \cdot 2\pi \cdot \left(-\cos\theta + \frac{2}{3}\cos^3\theta\right)\Big|_0^{\pi/2} =$$

$$= \frac{64}{5}\pi\left(1 - \frac{2}{3}\right) = \frac{64}{15}\pi$$