1.18 Låt $z = \ln(e^x + e^y)$.

(a) Visa att $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 1$

Obs $\frac{\partial z}{\partial x} = z'_x = \frac{1}{e^x+e^y} \cdot (e^x+e^y)'_x = \frac{e^x}{e^x+e^y}$

Analogt, $\frac{\partial z}{\partial y} = z'_y = \frac{e^y}{e^x+e^y}$

Notera att $\frac{e^x}{e^x+e^y} + \frac{e^y}{e^x+e^y} = 1$.

(b) Visa att $z''_{xx} \cdot z''_{yy} - (z''_{xy})^2 = 0$

Obs $z''_{xx} = (z'_x)'_x = \left(\frac{e^x}{e^x+e^y}\right)'_x = /\text{kvotregeln}/ =$

$= \frac{e^x(e^x+e^y) - e^x \cdot e^x}{(e^x+e^y)^2} = \frac{e^x \cdot e^y}{(e^x+e^y)^2}$

Analogt, $z''_{yy} = (z'_y)'_y = \frac{e^x \cdot e^y}{(e^x+e^y)^2}$.

$z''_{xy} = (z'_x)'_y = \left(\frac{e^x}{e^x+e^y}\right)'_y = \frac{0 - e^x \cdot e^y}{(e^x+e^y)^2}$

$= \frac{-e^x \cdot e^y}{(e^x+e^y)^2}$. Insats ger:

$\frac{e^x \cdot e^y}{(e^x+e^y)^2} \cdot \frac{e^x \cdot e^y}{(e^x+e^y)^2} - \left(\frac{-e^x \cdot e^y}{(e^x+e^y)^2}\right)^2 = 0.$

1.19 $\qquad F(x,y,t)=\frac{1}{t}e^{-(x^2+y^2)/at}$

Bestäm a s.a. F uppfyller ekv

$$\frac{\partial^2 F}{\partial x^2}+\frac{\partial^2 F}{\partial y^2}=\frac{\partial F}{\partial t}$$

Finn F''_{xx}: $\quad F'_x=\frac{1}{t}e^{-(x^2+y^2)/at}\cdot\left(\frac{-(x^2+y^2)}{at}\right)'_x=$

$$=\frac{-2x}{at^2}\cdot e^{-(x^2+y^2)/at}.$$

$$F''_{xx}=(F'_x)'_x=/\text{produktregeln}/=$$

$$\left(\frac{-2}{at^2}\right)\cdot e^{-(x^2+y^2)/at}-\frac{2x}{at^2}\cdot e^{-(x^2+y^2)/at}\cdot\left(\frac{-2x}{at}\right)$$

$$=\left(\frac{-2}{at^2}\right)\cdot e^{-(x^2+y^2)/at}\cdot\left(1-\frac{2x^2}{at}\right)$$

Analogt, $F''_{yy}=\left(-\frac{2}{at^2}\right)\cdot e^{-(x^2+y^2)/at}\cdot\left(1-\frac{2y^2}{at}\right)$

$$F'_t=\left(\frac{e^{-(x^2+y^2)/at}}{t}\right)'_t=/\text{kvotregeln}/=$$

$$=\frac{e^{'''}\cdot\frac{(x^2+y^2)}{at^2}\cdot t-e^{'''}\cdot 1}{t^2}=$$

$$e^{-(x^2+y^2)/at}\cdot\frac{1}{at^3}\cdot\left(x^2+y^2-at\right)=H.L.$$

Insats i ekv ger:

$$\text{V.L.} = \left(-\frac{2}{at^2}\right)e^{-(x^2+y^2)/at} \cdot \left(2 - \frac{2(x^2+y^2)}{at}\right) =$$

$$= \frac{4}{a^2t^3} e^{-(x^2+y^2)/at} \cdot (x^2+y^2-at)$$

$\underline{\text{V.L.} = \text{H.L.}} \Rightarrow \underline{a = 4}$

<u>1.20</u> Visa att det inte kan finnas

lösningar till systemet $\begin{cases} f'_x = x + 3yx^2 & (1) \\ f'_y = x^3 + xy & (2) \end{cases}$

Integrera (1) m a p x:

$$f = \int f'_x\,dx = \int (x + 3yx^2)\,dx = \frac{x^2}{2} + yx^3 + c(y)$$

Derivera f m a p y:

$$f'_y = \left(\frac{x^2}{2} + yx^3 + c(y)\right)'_y = x^3 + c'(y)$$

$(2) \Rightarrow \quad x^3 + c'(y) = x^3 + xy \iff c'(y) = xy$

men V.L. beror bara på y o

H.L. beror både på x o y.

$\Rightarrow$ det finns ej f som satisfierar (1) o (2)

Alternativt

Obs Varje lsg f till systemet är av klass $C^{(2)}$

Det betyder att $f''_{xy} = f''_{yx}$

$(1) \Rightarrow f''_{xy} = (f'_x)'_y = (x + 3yx^2)'_y = 3x^2$

$(2) \Rightarrow f''_{yx} = (f'_y)'_x = (x^3 + xy)'_x = 3x^2 + y$

Ty $3x^2 \neq 3x^2 + y \Rightarrow$ lsgar saknas.

1.21 Lös systemet $\begin{cases} f'_x = 2x + y & (1) \\ f'_y = 2y + x & (2) \end{cases}$

Integrera (1) m a p x:

$$f = \int f'_x \, dx = \int (2x + y) dx = x^2 + yx + c(y)$$

Obs!

Derivera f m a p y:

$f'_y = (x^2 + yx + c(y))'_y = x + c'(y)$

$(2) \Rightarrow \quad x + c'(y) = 2y + x \Leftrightarrow c'(y) = 2y \; (3)$

Integrera (3):

$c(y) = \int c'(y) dy = \int 2y \, dy = y^2 + d$, d är

en godtycklig konstant $\Rightarrow$

$f(x,y) = x^2 + yx + y^2 + d$

1.22 Bestäm f s.a

(a) $f_x' = 2x\sin x^2$, $f_y' = \cos y$

Gör som ovan:

(i) $f = \int f_x' \, dx = \int 2x \sin x^2 \, dx = \left/ \begin{matrix} t = x^2 \\ 2x\,dx = dt \end{matrix} \right/$

$= \int \sin t \, dt = -\cos x^2 + c(y)$

(ii) $f_y' = (-\cos x^2 + c(y))_y' = c'(y)$

Notera att $c'(y) = \cos y$

$\Rightarrow c(y) = \int c'(y)\,dy = \int \cos y \, dy = \sin y + d$

$\Rightarrow f(x,y) = -\cos x^2 + \sin y + d$, där d är godtycklig konstant.

(b) $f_x' = \dfrac{y}{x^2+y^2}$, $f_y' = -\dfrac{x}{x^2+y^2}$

(i) $f = \int f_x' \, dx = \int \dfrac{y}{x^2+y^2} \, dx = \int \dfrac{1}{y} \dfrac{dx}{\left(1+\left(\frac{x}{y}\right)^2\right)}$

$= \left/ \begin{matrix} t = \frac{x}{y} \\ dt = \frac{1}{y}dx \end{matrix} \right/ = \int \dfrac{dt}{1+t^2} = \arctan \dfrac{x}{y} + c(y)$

(ii) $f_y' = \dfrac{1}{1+\left(\frac{x}{y}\right)^2} \cdot \left(-\dfrac{x}{y^2}\right) + c' = \dfrac{-x}{x^2+y^2} + c'(y)$

Notera att $c'(y) = 0$

$\Rightarrow c(y) = \int 0 \, dy = d$ (godtycklig konstant)

$\Rightarrow f(x,y) = \arctan \frac{x}{y} + d$