2.17 Variabelbyte $\begin{cases} \xi = x + c\cdot t \\ \eta = x - c\cdot t \end{cases}$

Bestäm alla lsgar till ekv: $u''_{xx} - \frac{1}{c^2}\cdot u''_{tt} = 0$

$\underline{u(x,t) = ?}$

Obs $\xi'_x = 1, \ \xi'_t = c$

$\eta'_x = 1, \ \eta'_t = -c$

Använd Kedjeregeln:

$u'_x = u'_\xi \cdot \xi'_x + u'_\eta \cdot \eta'_x = u'_\xi + u'_\eta \quad (*)$

$u'_t = u'_\xi \cdot \xi'_t + u'_\eta \cdot \eta'_t = u'_\xi \cdot c - u'_\eta \cdot c \quad (**)$

$u''_{xx} = (u'_x)'_x \overset{(*)}{=} (u'_\xi + u'_\eta)'_x = (u'_\xi)'_x + (u'_\eta)'_x =$

$= /\text{Kedjeregeln}/ = \left((u'_\xi)'_\xi \cdot \xi'_x + (u'_\xi)'_\eta \cdot \eta'_x\right) +$

$\left((u'_\eta)'_\xi \cdot \xi'_x + (u'_\eta)'_\eta \cdot \eta'_x\right) = u''_{\xi\xi} + u''_{\xi\eta}$

$+ u''_{\eta\xi} + u''_{\eta\eta} = /\underbrace{u''_{\xi\eta} = u''_{\eta\xi}}/ = u''_{\xi\xi} + 2u''_{\xi\eta} + u''_{\eta\eta}$

$u''_{tt} = (u'_t)'_t \overset{(**)}{=} (u'_\xi \cdot c - u'_\eta \cdot c)'_t = c\cdot\left((u'_\xi)'_t - (u'_\eta)'_t\right) =$

$= /\text{Kedjeregeln}/ = c\cdot\left(\underbrace{(u'_\xi)'_\xi \cdot \xi'_t + (u'_\xi)'_\eta \cdot \eta'_t} -\right.$

$$-\left(\left(u'_\eta\right)'_\xi \cdot \xi'_t + \left(u'_\eta\right)'_\eta \cdot \eta'_t\right)\Big) = c\cdot\Big(u''_{\xi\xi}\cdot c - u''_{\xi\eta}\cdot c$$

$$-\left(u''_{\eta\xi}\cdot c - u''_{\eta\eta}\cdot c\right)\Big) = c\cdot\left(u''_{\xi\xi}\cdot c - 2u''_{\xi\eta}\cdot c + u''_{\eta\eta}\cdot c\right)$$

$$= c^2\cdot\left(u''_{\xi\xi} - 2u''_{\xi\eta} + u''_{\eta\eta}\right).$$

Sätt funna uttryck i ekvationen:

$$\left(u''_{\xi\xi} + 2u''_{\xi\eta} + u''_{\eta\eta}\right) - \frac{1}{c^2}\cdot c^2\left(u''_{\xi\xi} - 2u''_{\xi\eta} + u''_{\eta\eta}\right) = 0$$

eller $4u''_{\xi\eta} = 0$ $(\#) \Leftrightarrow u''_{\xi\eta} = 0$

Integrera $(\#)$:

(i) $u'_\xi = \int u''_{\xi\eta}\, d\eta = \int 0\cdot d\eta = a(\xi)$

(ii) $u = \int u'_\xi\, d\xi = \int a(\xi)\cdot d\xi = A(\xi) + B(\eta)$

d.v.s. $u(\xi,\eta) = A(\xi) + B(\eta)$

$\Rightarrow u(x,y) = A(x+c\cdot t) + B(x-c\cdot t)$, där

$A(\cdot)$, $B(\cdot)$ är två gånger deriverbara en-variabel funktioner

<u>2.18</u> Transformera $\dfrac{\partial^2 f}{\partial x \partial y}$ m h a

variabelbyte $\begin{cases} u = x+y \\ v = x \cdot y \end{cases}$

<u>Obs</u> $f''_{yx} = (f'_y)'_x \quad \left.\begin{array}{l} u'_x = 1,\ u'_y = 1 \\ v'_x = y,\ v'_y = x \end{array}\right\}$

(i) $f'_y = /\text{Kedjeregeln}/ =$

$= f'_u \cdot u'_y + f'_v \cdot v'_y = f'_u + x \cdot f'_v$

(ii) $f''_{yx} = (f'_u + x \cdot f'_v)'_x = (f'_u)'_x + (x \cdot f'_v)'_x =$

$= \underbrace{(f'_u)'_u \cdot u'_x + (f'_u)'_v \cdot v'_x}_{\text{1:a termen}} + \underbrace{1 \cdot f'_v + x \cdot (f'_v)'_x}_{\text{produkt regeln, 2:a termen}} =$

$= \underline{f''_{uu} + f''_{uv} \cdot y} + f'_v + x(\underbrace{(f'_v)'_u \cdot u'_x +}_{\text{Kedjeregeln}}$

$\underline{+ (f'_v)'_v \cdot v'_x}) = f''_{uu} + f''_{uv} \cdot y + f'_v +$

$+ x(f''_{vu} + y \cdot f''_{vv}) = /f''_{uv} = f''_{vu}/ =$

$= f''_{uu} + (x+y) f''_{uv} + (x \cdot y) \cdot f''_{vv} + f'_v =$

$f''_{uu} + u \cdot f''_{uv} + v \cdot f''_{vv} + f'_v$

2.21) Transformera $U = u''_{xx} + 2u''_{xy} + u''_{yy}$

genom att införa variablerna $\begin{cases} \xi = x^2 + y \\ \eta = 2x \end{cases}$

Obs $\xi'_x = 2x$, $\eta'_x = 2$

$\xi'_y = 1$, $\eta'_y = 0$

$$\left.\begin{aligned} u'_x &= u'_\xi \cdot \xi'_x + u'_\eta \cdot \eta'_x = u'_\xi \cdot 2x + u'_\eta \cdot 2 \\ u'_y &= u'_\xi \cdot \xi'_y + u'_\eta \cdot \eta'_y = u'_\xi \end{aligned}\right] \quad (*)$$

Använd (*) o kedjeregeln:

$$u''_{xx} = (u'_x)'_x = \left(2\left(u'_\xi \cdot x + u'_\eta\right)\right)'_x =$$

$$= 2\left((u'_\xi \cdot x)'_x + (u'_\eta)'_x\right) = 2\Big(\underbrace{(u'_\xi)'_x \cdot x + u'_\xi \cdot 1}_{\substack{\text{produktregeln} \\ \text{1:a termen}}} +$$

$$+ \underbrace{(u'_\eta)'_\xi \cdot \xi'_x + (u'_\eta)'_\eta \cdot \eta'_x}_{\text{2:a termen}}\Big) = 2\left(x \cdot \left((u'_\xi)'_\xi \cdot \xi'_x + (u'_\xi)'_\eta \cdot \eta'_x\right)\right.$$

$$\left. + u'_\xi + u''_{\eta\xi} \cdot 2x + u''_{\eta\eta} \cdot 2\right) = 2\left(x \cdot \left(u''_{\xi\xi} \cdot 2x + u''_{\xi\eta} \cdot 2\right)\right.$$

$$\left. + u'_\xi + u''_{\eta\xi} \cdot 2x + u''_{\eta\eta} \cdot 2\right) = 2\left(2x^2 u''_{\xi\xi} + 4x u''_{\xi\eta} +\right.$$

$$\left. + 2u''_{\eta\eta} + u'_\xi\right).$$

$$u''_{xy} = (u'_x)'_y = \left(2\cdot(u'_\xi \cdot x + u'_\eta)\right)'_y =$$

$$= 2\cdot\left((u'_\xi \cdot x)'_y + (u'_\eta)'_y\right) = 2\left((u'_\xi)'_y \cdot x + 0 +\right.$$

$$\left.+ (u'_\eta)'_y\right) = 2\left(\left((u'_\xi)'_\xi \cdot \xi'_y + (u'_\xi)'_\eta \cdot \eta'_y\right)\cdot x +\right.$$

$$\left.+ (u'_\eta)'_\xi \cdot \xi'_y + (u'_\eta)'_\eta \cdot \eta'_y\right) = 2\left(u''_{\xi\xi}\cdot x +\right.$$

$$\left.+ u''_{\xi\eta}\cdot 1\right)$$

$$u''_{yy} = (u'_y)'_y = (u'_\xi)'_y = (u'_\xi)'_\xi \cdot \xi'_y + (u'_\xi)'_\eta \cdot \eta'_y =$$

$$= u''_{\xi\xi}.$$

Sätt in funna uttryck i U:

$$U = 2\left(2x^2\cdot u''_{\xi\xi} + 4x\cdot u''_{\xi\eta} + 2u''_{\eta\eta} + u'_\xi\right)$$

$$+ 2\cdot 2\left(u''_{\xi\xi}\cdot x + u''_{\xi\eta}\right) + u''_{\xi\xi} =$$

$$= u''_{\xi\xi}\left(4x^2 + 4x + 1\right) + u''_{\xi\eta}\cdot\left(8x + 4\right) +$$

$$+ u''_{\eta\eta}\cdot 4 + 2u'_\xi = u''_{\xi\xi}\cdot(2x+1)^2 + u''_{\xi\eta}\cdot 4(2x+1)$$

$$+ u''_{\eta\eta}\cdot 4 + 2u'_\xi = u''_{\xi\xi}\cdot(\eta+1)^2 + u''_{\xi\eta}\cdot 4(\eta+1) +$$

$$+ u''_{\eta\eta}\cdot 4 + 2u'_\xi.$$