3.1 Beräkna gradienten till

$f(x,y)$: $\operatorname{grad} f = (\nabla f) = (f'_x, f'_y)$

(a) $f(x,y) = (x^2+y^2)^n$

$f'_x = n \cdot (x^2+y^2)^{n-1} \cdot (x^2+y^2)'_x = 2 \cdot x \cdot n \cdot (x^2+y^2)^{n-1}$

$f'_y = 2 \cdot y \cdot n \cdot (x^2+y^2)^{n-1} \Rightarrow$

$\nabla f = (2xn(x^2+y^2)^{n-1}, 2yn(x^2+y^2)^{n-1}) =$

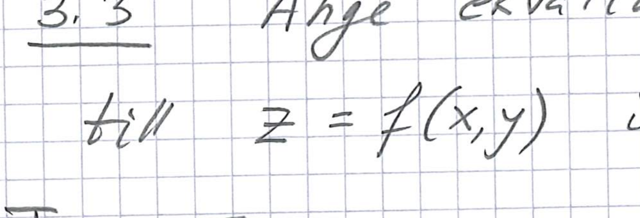
$= 2n(x^2+y^2)^{n-1} \cdot (x,y)$

(b) $f(x,y,z) = e^{xyz}$

$f'_x = e^{xyz} \cdot yz$, $f'_y = e^{xyz} \cdot xz$, $f'_z = e^{xyz} \cdot xy$

$\Rightarrow \nabla f = e^{xyz}(yz, xz, xy)$

3.3 Ange ekvationen för tangentplanet T till $z = f(x,y)$ i punkten $P(a,b,c)$



Obs $c = f(a,b)$

(ty P hör till grafen)

Tangentplanets ekvation:

$$\boxed{z = c + f'_x(a,b) \cdot (x-a) + f'_y(a,b) \cdot (y-b)}$$

Data: $z = \arcsin(xy)$, $P(1, \frac{1}{2}, \frac{\pi}{6})$

Gör kontroll: $\arcsin(1 \cdot \frac{1}{2}) = \arcsin \frac{1}{2} = \frac{\pi}{6}$ o.k.

$$f_x' = \frac{1}{\sqrt{1-(xy)^2}} \cdot (xy)_x' = \frac{1}{\sqrt{1-(xy)^2}} \cdot y$$

$$\left(\arcsin t' = \frac{1}{\sqrt{1-t^2}}\right)$$

$$f_y' = \frac{x}{\sqrt{1-(xy)^2}}$$ (analogt som ovan)

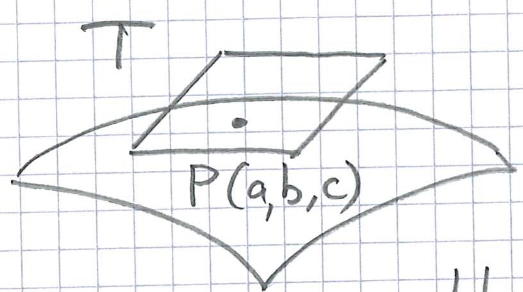
$$f_x'(1, \tfrac{1}{2}) = \frac{\frac{1}{2}}{\sqrt{1-\frac{1}{4}}} = \frac{1}{\sqrt{3}},$$

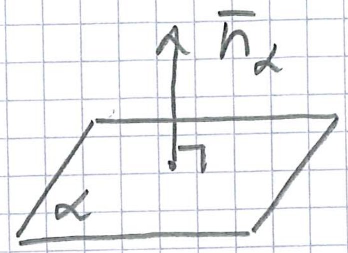
$$f_y'(1, \tfrac{1}{2}) = \frac{1}{\sqrt{1-\frac{1}{4}}} = \frac{2}{\sqrt{3}},$$

T: $z = \frac{\pi}{6} + \frac{1}{\sqrt{3}}(x-1) + \frac{2}{\sqrt{3}}(y - \frac{1}{2})$ eller

$$\frac{1}{\sqrt{3}}x + \frac{2}{\sqrt{3}}y - z + \frac{\pi}{6} - \frac{2}{\sqrt{3}} = 0$$

3.5. Finn alla punkter på ytan $x^2 + 2y^2 + 3z^2 + 2xy + 2yz = 1$, i vilka tangentplanet är parallellt med planet $x + y + z = 0$.





$F(x,y,z)=c$

bl.a. $F(P)=c$

ett plan

Inför $\begin{cases} F = x^2+2y^2+3z^2+2xy+2yz \\ c=1 \end{cases}$

Obs

$\alpha: \; x+y+z=0$, $\quad \bar{n}_\alpha(1,1,1) \perp \alpha$

Repetera att $\nabla F(P) \perp$ ytan $\Leftrightarrow$

$\nabla F(P) \perp T$

Vi söker P s.a.

$$\begin{cases} (1) \quad F(P)=1 \\ (2) \quad T \parallel \alpha \Leftrightarrow \nabla F(P) \parallel \bar{n}_\alpha \end{cases} \quad (*)$$

$$(*) \Leftrightarrow \begin{cases} F(P)=1 \\ \nabla F(P) = k\cdot \bar{n}_\alpha \quad \text{(det finns ett tal } k) \end{cases} \quad (**)$$

Stoppa i a, b, c i (**):

$F'_x = 2x+2y, \quad F'_y = 4y+2x+2z, \quad F'_z = 6z+2y$

$$(**) = \begin{cases} a^2+2b^2+3c^2+2ab+2bc=1 \\ (2a+2b,\; 2a+4b+2c,\; 2b+6c) = k\cdot(1,1,1) \end{cases}$$

eller $\begin{cases} a^2+2b^2+3c^2+2ab+2bc=1 & (\#) \\ 2a+2b=k \\ 2a+4b+2c=k \\ 2b+6c=k \end{cases}$ uttryck a, b, c i k $\underline{o}$

sätt in formlerna i (#)

$\Rightarrow a=\frac{3}{4}k, \quad b=-\frac{k}{4}, \quad c=\frac{k}{4} \quad (\#\#)$

$\underline{o}$ insättning:

$$\left(\frac{3}{4}k\right)^2+2\left(-\frac{k}{4}\right)^2+3\left(\frac{k}{4}\right)^2+2\cdot\frac{3}{4}k\cdot\left(-\frac{k}{4}\right)+$$

$$+2\left(-\frac{k}{4}\right)\cdot\frac{k}{4}=1 \quad \text{eller} \quad k^2\left(\frac{9}{16}+\frac{1}{8}+\frac{3}{16}-\frac{6}{16}\right.$$

$$\left.-\frac{2}{16}\right)=1 \quad \text{eller} \quad k^2\cdot\frac{3}{8}=1 \Rightarrow k=\pm\sqrt{\frac{8}{3}}$$

Finn punkterna genom att stoppa funna värde på k i (##):

$$P_1\left(\frac{3}{4}\sqrt{\frac{8}{3}}, -\frac{1}{4}\sqrt{\frac{8}{3}}, \frac{1}{4}\sqrt{\frac{8}{3}}\right) \quad \underline{o}$$

$$P_2\left(-\frac{3}{4}\sqrt{\frac{8}{3}}, \frac{1}{4}\sqrt{\frac{8}{3}}, -\frac{1}{4}\sqrt{\frac{8}{3}}\right) \quad \text{eller}$$

$$P_1, P_2 = \pm\frac{1}{\sqrt{6}}(3,-1,1)$$

<u>3.6</u> Bestäm ekv för tangentplanet

till ytan
$$(*)\quad \begin{cases} x = 2\cos u \sin v \\ y = 4\cos u \cos v \\ z = 2\sin u \end{cases}$$
i punkten P

som svarar mot $u = v = \frac{\pi}{4}$

Först, finn punkten P:

$$\left.\begin{array}{l} 2\cos\frac{\pi}{4}\sin\frac{\pi}{4} = 2\cdot\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}} = 1 \\ 4\cos\frac{\pi}{4}\cos\frac{\pi}{4} = 4\cdot\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}} = 2 \\ 2\sin\frac{\pi}{4} = 2\cdot\frac{1}{\sqrt{2}} = \sqrt{2} \end{array}\right]$$
d v s $P(1, 2, \sqrt{2})$

<u>Obs</u>
$$\begin{cases} \frac{x}{2} = \cos u \sin v \\ \frac{y}{4} = \cos u \cos v \\ \frac{z}{2} = \sin u \end{cases}$$

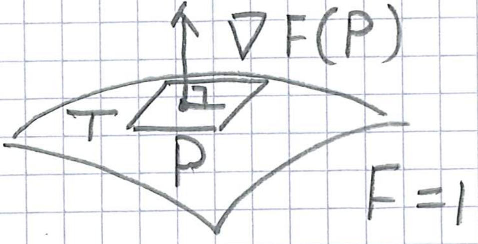
Kvadrera båda leden <u>o</u> addera!

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{4}\right)^2 + \left(\frac{z}{2}\right)^2 = (\cos u \sin v)^2 + (\cos u \cos v)^2 + (\sin u)^2$$
eller

$$\boxed{\frac{x^2}{4} + \frac{y^2}{16} + \frac{z^2}{4} = 1}$$

(använd trigonometriska ettan!)

Inför $F(x,y,z) = \frac{x^2}{4} + \frac{y^2}{16} + \frac{z^2}{4}$

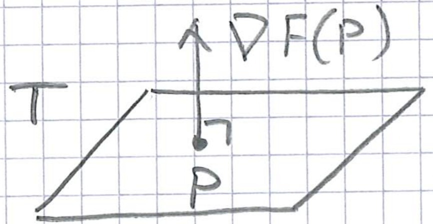


är en del av ytan (*) runt P.

Obs $\nabla F(P) \perp$ ytan ($\Leftrightarrow \nabla F(P) \perp T$)

Finn $\nabla F(P)$:

$F'_x = \frac{x}{2}$, $F'_y = \frac{y}{8}$, $F'_z = \frac{z}{2}$, $\nabla F(P) = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{\sqrt{2}}\right)$



$T: \frac{1}{2}(x-1) + \frac{1}{4}(y-2) + \frac{1}{\sqrt{2}}(z-\sqrt{2}) = 0$

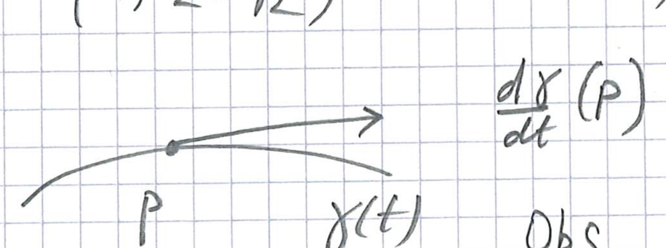
eller $\frac{1}{2}x + \frac{1}{4}y + \frac{1}{\sqrt{2}}z - 2 = 0$

3.8 (i) Visa att kurvan $\gamma(t) = (\sin 2t, \sin^2 t, \cos t)$, $t \in [0, 2\pi]$ ligger på ytan $x^2 + 4y^2 + 4z^2 = 4$:

$\sin^2 2t + 4\sin^4 t + 4\cos^2 t = \underline{4\sin^2 t\cos^2 t + 4\sin^4 t}$
$+ 4\cos^2 t = 4\sin^2 t(\underbrace{\cos^2 t + \sin^2 t}_{=1}) + 4\cos^2 t = 4$

v. h. v.

(ii) Visa att tangenten till γ i punkten $P(1, \frac{1}{2}, \frac{1}{\sqrt{2}}) \perp \nabla F(P)$, där $F = x^2 + 4y^2 + 4z^2$:



Obs P svarar mot $t = \frac{\pi}{4}$

$$\frac{d\gamma}{dt} = \left((\sin 2t)', (\sin^2 t)', (\cos t)' \right) =$$

$$= (2\cos 2t, \ 2\sin t \cos t, \ -\sin t) \Rightarrow \frac{d\gamma}{dt}(P) =$$

$$\left(2\cos\left(2\cdot\frac{\pi}{4}\right), \ \sin\left(2\cdot\frac{\pi}{4}\right), \ -\sin\frac{\pi}{4}\right) = \left(0, 1, -\frac{1}{\sqrt{2}}\right)$$

$$\nabla F = (F'_x, F'_y, F'_z) = (2x, 8y, 8z) \Rightarrow$$

$$\nabla F(P) = (2, 4, 4\sqrt{2})$$

Kolla $\nabla F(P) \cdot \frac{d\gamma}{dt}(P) = (2, 4, 4\sqrt{2}) \cdot \left(0, 1, -\frac{1}{\sqrt{2}}\right)$

skalärprodukt $= 0 + 4 - 4 = 0$

$\Rightarrow \nabla F(P) \perp \frac{d\gamma}{dt}(P)$ v.s.v.

<u>3.10</u> Riktningsderivata $f'_{\bar{v}}(P) =$

$= \nabla f(P) \cdot \frac{\bar{v}}{|\bar{v}|}$ (skalärprodukt)

data: $\begin{cases} f(x,y,z) = xyz, & \nabla f(P) = (yz, xz, xy), \quad P(x,y,z) \\ \bar{v} = (1,2,2), & |\bar{v}| = \sqrt{1+2^2+2^2} = 3 \end{cases}$

$$\Rightarrow f'_{\bar{v}}(P) = (yz, xz, xy) \cdot \frac{1}{3}(1,2,2) =$$

$$= \frac{1}{3}(yz + 2xz + 2xy)$$

3.11 $\quad f(x,y,z) = xy + e^{yz} + z$

$\bar{v}(\alpha,\beta,\gamma)$ s.a. $\alpha^2+\beta^2+\gamma^2=1$

Finn riktningsderivata $f'_{\bar{v}}$:

$\nabla f = (y,\ x + e^{yz}\cdot z,\ e^{yz}\cdot y + 1)$, Obs $|\bar{v}|=1$

$\Rightarrow f'_{\bar{v}} = \nabla f \cdot \frac{\bar{v}}{|\bar{v}|} = \alpha y + \beta(x+e^{yz}\cdot z) + \gamma(e^{yz}\cdot y+1)$.

Hur skall (α,β,γ) väljas för att derivatan skall bli så stor som möjligt?

Svar: $\bar{v} = \frac{\nabla f}{|\nabla f|}$;