Solutions

TAMS24/TEN1 2019-09-07

1. (a) Let $X_i \sim \mathrm{N}(\mu_X, \sigma^2)$ be the new random sample. It follows that (by Cochran's and Gosset's theorems)

$$T_X = \frac{\overline{X} - \mu_X}{S/\sqrt{8}} \sim t(7),$$

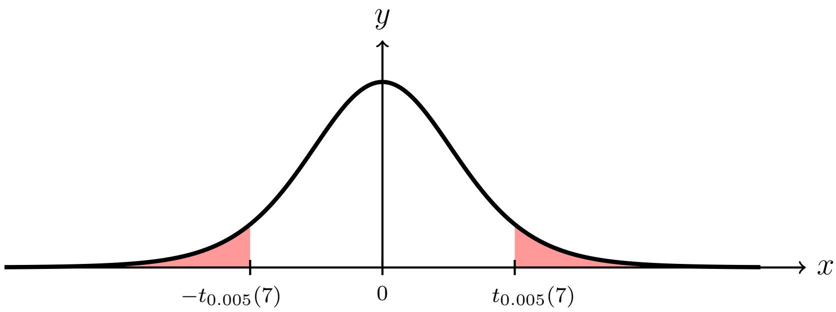
and

$$P(-t_{\alpha/2}(7) < T_X < t_{\alpha/2}(7)) = 1 - \alpha,$$

where we can solve the inequality for

$$\overline{X} - t_{\alpha/2}(7) \cdot \frac{S}{\sqrt{8}} < \mu_X < \overline{X} + t_{\alpha/2}(7) \cdot \frac{S}{\sqrt{8}}.$$

From a table, we find that $t_{0.005}(7) = 3.50$.



As an observation of S_X, we use $\sqrt{s_X^2}$, so

$$t_{0.005}(7)\frac{s}{\sqrt{8}} = 3.50 \cdot \frac{0.5032}{2.8284} = 0.6226.$$

Since $\overline{x} = 6.965$, the interval is given by

$$I_{\mu_X} = (6.34,\ 7.59).$$

(b) Let $Y \sim \mathrm{N}(\mu_Y, \sigma^2)$ be the old sample. Since the variances are equal, we weight them together according to the pooled variance:

$$s^2 = \frac{7s_1^2 + 7s_2^2}{14} = \frac{1}{2}\left(s_1^2 + s_2^2\right).$$

It now follows that (by Cochran's and Gosset's theorems)

$$T = \frac{0.5\overline{X} - \overline{Y} - (0.5\mu_X - \mu_Y)}{S\sqrt{\frac{0.5^2}{8} + \frac{1}{8}}} \sim t(16-2) = t(14),$$

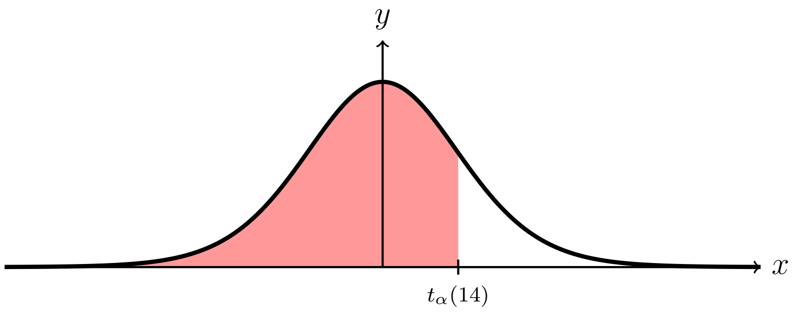
and

$$P(T < t_\alpha(14)) = 1 - \alpha,$$

where we can solve the inequality for

$$0.5 \cdot \overline{X} - \overline{Y} - t_\alpha(14) \cdot S\sqrt{\frac{0.5^2}{8} + \frac{1}{8}} < 0.5 \cdot \mu_X - \mu_Y.$$

We use a one-sided interval since we only want to investigate if $0.5 \cdot \mu_X > \mu_Y$. From a table, we find that $t_{0.05}(14) = 1.7613$.



As an observation of S, we use $\sqrt{s^2}$, so

$$t_{0.05}(14)\cdot s\cdot\sqrt{\frac{0.5^2}{8}+\frac{1}{8}} = 1.7613\cdot 0.4520\cdot 0.3953 = 0.3147.$$

Since $0.5\cdot\overline{x}-\overline{y} = 0.3312$, the interval is given by

$$\begin{aligned} I_{0.5\cdot\mu_X-\mu_Y} &= (0.3312 - 0.3147,\, \infty) \\ &= (0.0165,\, \infty). \end{aligned}$$

We see that 0 is not included in the interval, so we can claim that it is likely that $0.5\cdot\mu_X > \mu_Y$ at this significance level.

(c) Let

$$H_0 : \sigma_X^2 = \sigma_Y^2 = \sigma^2$$

and

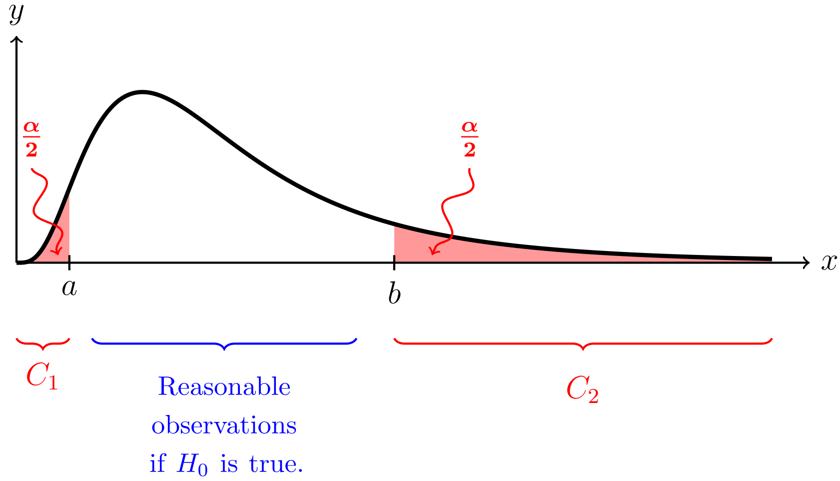
$$H_1 : \sigma_X^2 \neq \sigma_Y^2.$$

If H_0 is true, then $7S_X^2/\sigma^2 \sim \chi^2(7)$ and $7S_Y^2/\sigma^2 \sim \chi^2(7)$. Thus

$$V = \frac{\dfrac{7S_X^2}{\sigma^2}/7}{\dfrac{7S_Y^2}{\sigma^2}/7} = \frac{S_X^2}{S_Y^2} \sim F(7,7)$$

since S_1^2 and S_2^2 are independent. We seek a critical region C such that

$$\alpha = P(V\in C\,|\,H_0).$$



We find the bounds a and b from a table so that

$$P(V < a) = P(V > b) = \frac{\alpha}{2}.$$

with $a = 0.2002$ and $b = 5.00$. Note that a two-sided interval is necessary here. Since

$$v = \frac{0.2532}{0.1553} = 1.6306 \notin C$$

we can't reject H_0. The variances could be equal (but are they?)

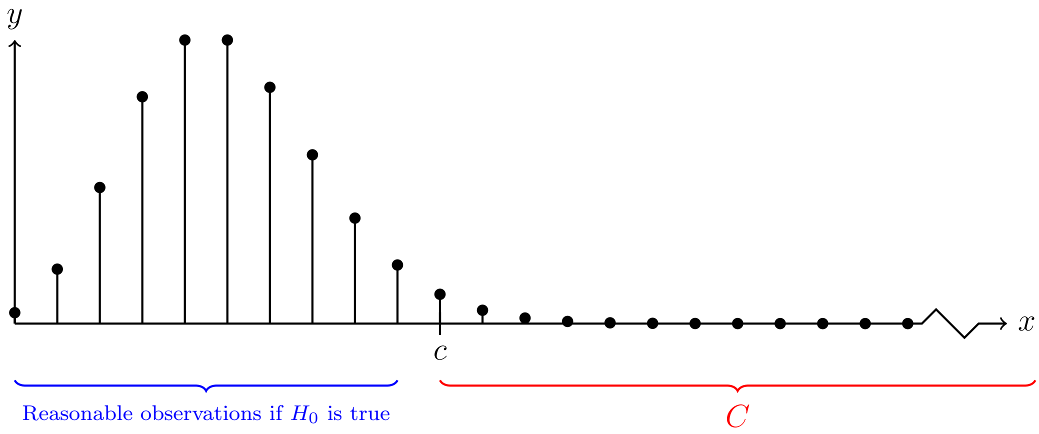
Answer:

(a) $(6.34,\ 7.59)$

(b) The new expectations seems to be more than twice the old one.

(c) We can't reject the hypothesis that they are equal; we do not know.

2. (a) Assume that H_0 is true. Let $X \sim \text{Po}(5)$ (since the expected number of counts is 5 during 1 second). We need to find the critical region C.



Let $p(k)$, $k = 0, 1, 2, \ldots$, be the probability function for X. From a table we can find that

$$\sum_{k=9}^{\infty} p(k) = 1 - \sum_{k=0}^{8} p(k) = 0.0681 \quad \text{and} \quad \sum_{k=10}^{\infty} p(k) = 0.0318.$$

Thus we choose

$$C = \{k \in \mathbf{Z} : k \geq 10\}.$$

Since our observation is $x = 8$ and $8 \notin C$, we can't reject H_0. It is possible that $\mu = 5$. Great news, right?!

(b) The power at $\mu = 10$ can be calculated straight from the definition:

$$h(10) = P(H_0 \text{ rejected} \,|\, \mu = 10) = P(X \in C \,|\, \mu = 10)$$
$$= \sum_{x=10}^{\infty} e^{-10} \frac{10^x}{x!} = 0.5421.$$

Answer: (a) We can't reject H_0. It could be that $\mu = 5$ (we do not know). (b) 0.5421.

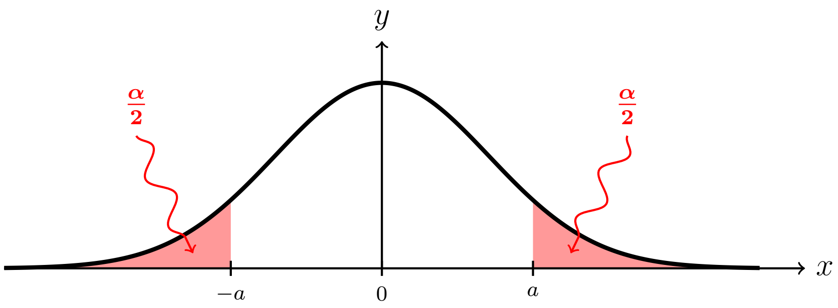
3. When measuring for 10 seconds, the expectation of $Y = X(10) \sim \text{Po}(10\cdot\lambda)$ is $E(Y) = 10\cdot\lambda$. We have observed $y = 82$, so it is reasonable to assume that $E(Y) > 15$. Thus we can use a normal approximation for Y. Moreover, $V(Y) = 10\cdot\lambda$ (the Poisson distribution is funny..). Thus

$$Z = \frac{Y - 10\lambda}{\sqrt{10\widehat{\lambda}}} \overset{\text{appr.}}{\sim} N(0,1),$$

so we find $a > 0$ so that

$$0.90 = P(-a < Z < a).$$

Here we'll use the estimate $\widehat{\lambda} = 8.2$ (this will simplify matters).



So $a = 1.645$ is suitable. Then

$$-a < Z < a \quad\Leftrightarrow\quad -a < \frac{Y - 10\lambda}{\sqrt{10\widehat{\lambda}}} < a \quad\Leftrightarrow\quad \frac{Y - a\sqrt{10\widehat{\lambda}}}{10} < \lambda < \frac{Y + a\sqrt{10\widehat{\lambda}}}{10}$$

so with the observation $y = 82$ and estimate $\widehat{\lambda} = 8.2$, we obtain the (approximate) confidence interval

$$I_\lambda = (6.7, 9.9).$$

Answer: $I_\lambda = (7.04, 9.81)$.

4. (a) Let $u = (1\ 2\ 5\ 0)^T$ and let Y_0 be an independent random observation at $a = 2$, $r = 5$ and $h = 0$. Let $\widehat{\mu_0}$ be the estimate for the expectation μ at the same point. A well known test quantity is

$$T = \frac{Y_0 - \widehat{\mu_0}}{S\sqrt{1 + \boldsymbol{u}^T(X^TX)^{-1}\boldsymbol{u}}} \sim t(10-4) = t(6).$$

We can box in this variable and solve for Y_0:

$$-t < T < t \quad\Leftrightarrow\quad -t < \frac{Y_0 - \widehat{\mu_0}}{S\sqrt{1 + \boldsymbol{u}^T(X^TX)^{-1}\boldsymbol{u}}} < t$$
$$\Leftrightarrow\quad \widehat{\mu_0} - tS\sqrt{1 + \boldsymbol{u}^T(X^TX)^{-1}\boldsymbol{u}} < Y_0 < \widehat{\mu_0} + tS\sqrt{1 + \boldsymbol{u}^T(X^TX)^{-1}\boldsymbol{u}},$$

where $t = t_{\alpha/2}(6) = t_{0.005}(6) = 1.9432$. We can now calculate that

$$\boldsymbol{u}^T(X^TX)^{-1}\boldsymbol{u} = 0.412,$$

so $\sqrt{1 + \boldsymbol{u}^T(X^TX)^{-1}\boldsymbol{u}} = 1.1883$. As an observation of S, we use

$$s = \sqrt{\frac{\text{SS}_\text{E}}{10-4}} = \sqrt{\frac{4.35}{6}} = 0.725.$$

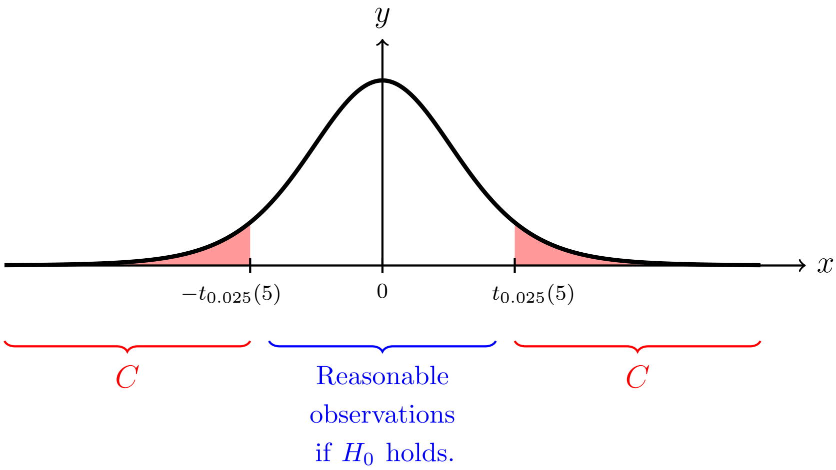
For $\widehat{\mu}_0$, we use the observation $\boldsymbol{u}^T\widehat{\boldsymbol{\beta}} = 15.6755$. Thus we obtain the prediction interval

$$I_{Y_0} = \Big(15.6755 \mp 1.9432 \cdot 0.725 \cdot 1.1883\Big) = (14.0,\ 17.35).$$

(b) To test if $\beta_3 = 0$, let $H_0 : \beta_3 = 0$ and $H_1 : \beta_3 \neq 0$. Assume that H_0 holds. Then

$$T = \frac{\widehat{\beta}_3 - 0}{S\sqrt{h_{33}}} \sim t(10-4) = t(6),$$

where the distribution is clear since H_0 holds. We need a critical region C such that $P(T \in C \mid H_0) = 0.01$ and since H_1 is double sided, we choose symmetrically.



We find $t_{\alpha/2}(6) = t_{0.005}(6) = 3.7074$ in a table. An observation of $S\sqrt{h_{33}}$ is given by the standard error $d(\widehat{\beta}_3)$ and thus we find that the observation

$$t = \frac{0.28}{0.56} = 0.5$$

does not belong to the critical region. So we can't reject H_0. The coefficient β_2 might be zero.

Answer:

(a) $(14.0,\ 17.35)$.

(b) A significance test shows that we can't conclude that $\beta_3 \neq 0$ at the significance level 1%. The addition of growth hormone might not have any effect.

5. (a) We start by noting that the parameter space is $\Omega_\theta = \{0, 1, 2, 3, 4, 5\}$. This means that continuous methods are problematic and it might be better to just find the ML-estimator directly. Why? Well, let us look at the likelihood function. Each X_i has the probability function

$$p_{X_i}(k) = \begin{cases} 1 - \frac{\theta}{5}, & k = 0, \\ \frac{\theta}{5}, & k = 1, \end{cases}$$

where the outcome $X_i = 1$ means that we've found a tentaclified flamingo. Thus the likelihood function is given by

$$L(\theta) = \prod_{i=1}^{n} p_{X_i}(x_k) = \left(1 - \frac{\theta}{5}\right)^{n - \sum x_k} \left(\frac{\theta}{5}\right)^{\sum x_k} = \left(1 - \frac{\theta}{5}\right)^{n(1-\overline{x})} \left(\frac{\theta}{5}\right)^{n\overline{x}}.$$

In our case we have $n = 7$, but we'll get to that. If we were to consider $L(\theta)$ as a function of a continuous $\theta > 0$, we could proceed as usual:

$$l(\theta) = \ln L(\theta) = n(1 - \overline{x}) \ln \frac{5 - \theta}{5} + n\overline{x} \ln \frac{\theta}{5},$$

so

$$0 = l'(\theta) = -\frac{n(1 - \overline{x})}{5 - \theta} + \frac{n\overline{x}}{\theta} \quad \Leftrightarrow \quad \theta = 5\overline{x}.$$

We can verify that $\widehat{\theta} = 5\overline{x}$ actually is a maximum of $l'(\theta)$ by observing the sign change of $l'(\theta)$, i.e., that we obtain $\nearrow$ max $\searrow$ for the function $l(\theta)$, so this would be our ML-estimate. However, there's no way to guarantee that $\widehat{\theta} \in \Omega_\theta$. We might surmise though that what we're looking for is close to $5\overline{x}$.

Using our sample $(x_1, \dots, x_6) = (1, 0, 1, 0, 1, 1, 0)$, we see that

$$L(\theta) = \left(1 - \frac{\theta}{5}\right)^3 \left(\frac{\theta}{5}\right)^4,$$

so doing the calculations we find that

$$L(\theta) = 10^{-3} \cdot \begin{cases} 0, & \theta = 0, \\ 0.8192, & \theta = 1, \\ 5.5296, & \theta = 2, \\ 8.2944, & \theta = 3, \\ 3.2768, & \theta = 4, \\ 0, & \theta = 5. \end{cases}$$

We see that $\theta = 3$ provides the highest probability, so by definition this is the MLE. What happens with the continuous version? Well, we have $5\overline{x} = 5 \cdot \frac{4}{7} = 2.857$. Rounding it we'd obtain $\widehat{\theta} = 3$, but we would have to prove that this is the actual MLE in that case.

(b) This is a rather tricky question. If you'd try and just use the estimator $5\overline{x}$ (which gives values very likely outside of the parameter space), it's rather easy to show that it is unbiased:

$$E(5\overline{X}) = 5E(X) = 5\frac{\theta}{5},$$

where X has the same distribution as all the X_i:s in the average. But we can't really use this as our estimator. So what about the integer part of $5\overline{X}$? Well, the maximum should (considering the investigation above) happen if we either round up or down, so how would we know which one? Should we choose whichever is closest? What's to say that choosing in that way would yield the true MLE? So yeah.. difficult to answer :).

Answer: (a) $\widehat{\theta} = 3$ (b) see above.

6. Since A is invertible, we know that $\det A \neq 0$.

$\Rightarrow$) This direction is more or less trivial. If the components of $\boldsymbol{Y}$ are independent, then $C(Y_i, Y_j) = 0$ for $i \neq j$ and $C(Y_i, Y_i) = \sigma_i^2$, so $C_{\boldsymbol{Y}}$ is obviously a diagonal matrix.

$\Leftarrow$) Now suppose that $C_{\boldsymbol{Y}}$ is a diagonal matrix, e.g.,

$$\begin{pmatrix} \sigma_1^2 & 0 & \cdots & 0 \\ 0 & \sigma_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_n^2 \end{pmatrix}.$$

Since $C_{\boldsymbol{Y}} = AA^T$ we know that $C_{\boldsymbol{Y}}$ is invertible, which implies that $\sigma_i^2 \neq 0$ for all i. The inverse $C_{\boldsymbol{Y}}^{-1}$ is the diagonal matrix with the diagonal elements σ_i^{-2}. We thus obtain the joint density function

$$\begin{aligned} f_{\boldsymbol{Y}}(\boldsymbol{y}) &= \frac{1}{(2\pi)^{n/2}\sqrt{\det C_{\boldsymbol{Y}}}} \exp\left(-\frac{1}{2}(\boldsymbol{y}-\boldsymbol{\mu})^T C^{-1}(\boldsymbol{y}-\boldsymbol{\mu})\right) \\ &= \frac{1}{(\sqrt{2\pi})^n \sigma_1\sigma_2\cdots\sigma_n} \exp\left(-\frac{1}{2}\sum_{j=1}^n (y_j-\mu_j)\sigma_j^{-2}(y_j-\mu_j)\right) \\ &= \prod_{j=1}^n \frac{1}{\sigma_j\sqrt{2\pi}} \exp\left(-\frac{(y_j-\mu_j)^2}{2\sigma_j^2}\right) = \prod_{j=1}^n f_{Y_j}(y_j). \end{aligned}$$

We have now shown that the joint density function is given by the product of the density functions for Y_j, which immediately implies that the components of $\boldsymbol{Y}$ are independent.

Answer: See above.